

Motion of test bodies with internal degrees of freedom in non-Euclidian spaces.

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December 24, 2009

Abstract

Discussed is mechanics of objects with internal degrees of freedom in generally non-Euclidean spaces. Geometric peculiarities of the model are investigated detailly. Discussed are also possible mechanical applications, e.g., in dynamics of structured continua, defect theory and in other fields of mechanics of deformable bodies. Elaborated is a new method of analysis based on non-holonomic frames. We compare our results and methods with those of other authors working in nonlinear dynamics (many of them refer to our papers [20], [21], [49], [50]). Simple examples of completely integrable models are presented.

Keywords: affine invariance, affinely-rigid bodies, collective modes, internal degrees of freedom, nonlinear elasticity, Riemannian manifolds.

1 Introduction

In our earlier papers [49], [50] (and reference therein) we discussed systems with collective and internal degrees of freedom ruled by the affine and linear groups, first of all the metric-preserving groups, i.e., the isometry group and its homogeneous part, i.e., the rotation group (orthogonal group). Roughly speaking, we were dealing there with rigid bodies, i.e., gyroscopes, and affinely-rigid bodies, i.e., homogeneously deformable gyroscopes, both in the flat Euclidean space. Such objects are interesting from the point of view of purely rational analytical mechanics in itself, and besides, they occur in various quite practical problems [12], [39], [40], [42], [43], [49], [50], [51]. In continuum mechanics they occur as models of internal degrees of freedom, i.e., microstructure models in mechanics of complex bodies. Classical examples are micropolar continua of brothers Cosserat and micromorphic continua of Eringen [14], [15], [16], [17], [19], [35], [36]. Roughly speaking, they are continua of infinitesimal gyroscopes or homogeneously deformable gyroscopes. They describe some granular media; another

application is the long-wave continuum limit of dynamics of molecular crystals. Such continua do not consist any longer of the Newton type material points; instead, their elementary constituents are material points with extra attached orthonormal or general bases (respectively the micropolar and micromorphic continua). Let us mention there is also plenty of other microstructure models where at material points some other geometric objects are attached like e.g. liquid crystals which are, roughly speaking, continua of infinitesimal rods (in continuous limit of course) [10], [11], [28], [62]. To understand properly the dynamics of generalized continua, one must start from the well-defined dynamics of extended affinely-rigid bodies or their systems in a flat Euclidean space.

Below we consider something more peculiar, namely the dynamics of very small, essentially infinitesimal rigid or affinely-rigid bodies in a curved Riemannian manifold, or even, more generally, in a manifold endowed only with affine connection. One must answer seriously the question concerning motivation. Such models are mathematically very nice and interesting on the level of purely rational analytical mechanics. But it is clear that we are living in physical space which in a very good approximation is Euclidean (in the scale of our every-day life generally-realistic effects are negligible). Of course, one can answer immediately that this may be an intermediary step towards constructing relativistic mechanics of continua [5], [26], [45], [54]. But topic, although interesting, is perhaps esoteric from the point of view of every-day praxis. Let us notice, however that there exist some problems where the motion of small objects in curved spaces becomes a practically viable topic. First of all let us observe that in a deformed elastic medium we are given two spatial metric tensors: the usual metric tensor g_{ij} of the physical space, which is in a sense the manifestation of the vacuum state of the gravitational field, and the Cauchy deformation tensor C_{ij} , [19], [31],

$$C_{ij} = \eta_{KL} \frac{\partial a^K}{\partial x^i} \frac{\partial a^L}{\partial x^j}. \quad (1)$$

In the above formula η denotes the metric tensor of the material space (the reference metric tensor), and a^K are Lagrange (reference) coordinates expressed as functions of the Euler (spatial) coordinates x^i . Perhaps more naturally looks the expression for the reciprocal contravariant tensor,

$$C^{ij} = \frac{\partial x^i}{\partial a^K} \frac{\partial x^j}{\partial a^L} \eta^{KL}, \quad C^{im} C_{mj} = \delta^i_j. \quad (2)$$

The point is that for small, concentrated objects in the deformed continuum, like, e.g., defects (dislocations, disclinations, vacancies, interstitials [3], [64]), it may be not g_{ij} but just C_{ij} that is felt as something like metric tensor. Similar situations have place in solid state physics, where due to certain collective phenomena, it is used not the usual, geometric metric tensor, but rather the tensor of effective mass which in certain situations even need not be positively definite. It is reasonable to expect something similar here. Of course, the space endowed with C_{ij} as the metric tensor is still flat due to the non-separability (compatibility) conditions [19], [54], because the curvature tensor built of C does vanish.

But it is well-known that the point defects feel the body manifold as curved Riemannian space. Similar concepts appear in theory of residual stresses.

Second, let us put our attention on two-dimensional surface phenomena. The surfaces of real bodies are curved submanifolds on dimension two. One can expect that for many surface phenomena it is just the two-dimensional induced metric that is relevant for the dynamics. For example, the microstructure of the bulk of the body may generate some effective microstructure on the boundary, and no doubt that for the corresponding surface phenomena the induced metric may be more relevant than the true metric tensor g_{ij} . Even without any microstructure, the induced two-dimensional metric seems to be an important factor, e.g., for the surface waves. Certainly it is so in dynamics of the boundary membranes (films) of biological cells.

Let us also stress some ecological problems, like the motion of pollution regions (spots) on the oceanic surface after damages of tankers. This is just the classical example of the motion of (relatively) small two-dimensional drop over the curved spherical surface. One can also think about the motion of continental plates and similar geophysical problems [20], [48], [49].

Of course, it is convenient to begin any analysis from the motion of bodies in constant curvature space, like the spherical or hyperbolic (Lobachevski) space, because one can expect rigorous analytical solutions, just in virtue of the existence of high symmetries. There are some advantages of beginning the analysis from the academic case of n -dimensional spaces restricting only then the considerations to physical values of n .

Within generally relativistic context the motion of extended structured bodies was studied by Mathisson, Weyssenhoff, Tulczyjew and Papapetru [32], [57], [37]; let us stress the contribution of the first three Polish physicists. On the non-relativistic level the problems of rigid motion in curved spaces were discussed by Yugoslavian mechanician Stojanović [56]. We follow general ideals of Künzle [26], [45] for whom such an object was described by linear frame attached to the material point. This is a mathematical idealization of very small objects when the linear size of the body is small in comparison with the size at which the spatial curvature changes remarkably. What we do is somehow related to the pole-dipole approximation due to Mathisson, Weyssenhoff, Tulczyjew and Papapetrou. It is reasonable to expect that performing some more detailed analysis one can obtain more adequate higher multipole description where equations of motion contain not only Riemann tensor but also its covariant derivatives. Therefore, the situation will be particularly simple in symmetric spaces (where the covariant derivative of the curvature tensor vanishes) [25]. The analysis probably would have to be lead with the help of normal coordinates and make use of certain asymptotic power series expansion.

Before going any further, let us begin with explaining why we use here the approach based on the concept of infinitesimal rigid or affinely rigid bodies, at least as a first approximation.

For extended systems of material points in a flat space, any two admissible configurations of the usual rigid body (gyroscope) are related to each other by some isometry transformation. The isometry group of the flat Euclidean space

of dimension n is a $\frac{1}{2}n(n+1)$ - dimensional Lie group. However, in a generic Riemann space the typical situation is that the isometry group has dimension zero and consists of merely identity transformations. In any case this dimension k always satisfies the inequality

$$k \leq \frac{1}{2}n(n+1) \quad (3)$$

and the maximal possible value is attained in constant-curvature-spaces, including of course the flat spaces as a special example [25]. But even in a non-flat constant curvature space there are some problems, namely the centre of mass concept is not correctly defined [46], [55], and because of this there is no splitting into translational and internal motion. It is so when dealing with extended bodies. So there is only one way to escape the problem: to concentrate the attention on sufficiently small bodies and go to the limit with their size. Then we obtain just the non-Newtonian material point with something attached to it. In our models this "something" is an orthonormal basis (infinitesimal gyroscope) or just a general basis (infinitesimal affinely-rigid body, i.e., an infinitesimal homogeneously deformable body). Then mathematically everything becomes correct, and physically one obtains a good approximate description of small metrically-rigid or affinely-rigid bodies.

To be complete with all these arguments, let us remind the concepts of isometries in Riemann spaces and affine transformations in spaces endowed with affine connection. Let (M, g) be an n -dimensional Riemannian space; M is a differential manifold and g the metric tensor on M . We say that the mapping φ of M onto M is an isometry if

$$g = \varphi^* g. \quad (4)$$

Analytically, when φ is given by the dependence of the new point coordinates y^i on the old ones x^i , this is given by

$$g_{ij} = g_{ab} \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j}. \quad (5)$$

Let now Γ be an affine connection, e.g., the Levi-Civita one built of g , but not necessarily, it may be a general connection, even completely non-related to g . Let ∇ denote the corresponding covariant differentiation, and ∇_X the corresponding directional covariant derivative along the vector field X . We say that the diffeomorphism φ of M onto M is an affine mapping if for any vector fields X, Y on M the following holds:

$$\nabla_{(\varphi_* X)} (\varphi_* Y) = \varphi_* (\nabla_X Y), \quad (6)$$

where φ_* is an abbreviation for the action of φ on vector fields (thus, the covariant differentiation is transparent with respect to the action of φ). Just like isometries, in a general manifold with connection, the existence of affine diffeomorphisms is rather exceptional.

2 Degrees of freedom, kinematics, phase space, symmetries

From now on the physical space M is a differential manifold endowed with a Riemannian structure, i.e., with some positively definite metric tensor field g , or, in certain situations, even with the weaker structure given by some affine connection Γ . One can also consider the double structure when geometry of M is given by some pair (g, Γ) where some relationship between g and Γ may exist or not. As stated in Introduction, we replace extended bodies by structured material points with attached linear bases. When those bases are by definition g - orthonormal, we deal with the infinitesimal gyroscope; if they are general, our system is an infinitesimal affinely rigid body (homogeneous deformable gyroscope). One can also consider intermediate situations when some weaker constraints are imposed onto infinitesimal affine motion (incompressibility etc.). The attached bases describe internal degrees of freedom and give the symbolic description of relative degrees of freedom after the limit transition, when the diameter of the body tends to zero. Let our object be instantaneously placed at the position $x \in M$ and some linear frame $e = (e_1, \dots, e_A, \dots, e_n)$ be attached at that point, thus

$$e_A \in T_x M, \quad A = 1, \dots, n. \quad (7)$$

For infinitesimal affinely rigid bodies those are general linear frames, for infinitesimal gyroscopes they are orthonormal with respect to the metric g ,

$$(e_A | e_B) = g(e_A, e_B) = g_{ij} e^i_A e^j_B = \eta_{AB}, \quad (8)$$

where η is a fixed reference metric in the micromaterial space N . Usually, but not always and not necessarily we put $N = R^n$ and take η to be the Kronecker delta,

$$\eta_{AB} = \delta_{AB}. \quad (9)$$

The small latin indices refer to the physical space as seen, whereas the capital ones refer to the micromaterial space N . Injections of the body into M , or rather into tangent spaces $T_x M$ are given by:

$$y^i(t, a) = x^i(t) + \varphi^i_K(t) a^K, \quad (10)$$

where x^i is the position of the material point, the quantities φ^i_K describe the relative/internal motion, and a^K are Lagrangian coordinates in the micromaterial space N . More precisely, the formula (10) is an approximation which is valid in virtue of the infinitesimal character of the body; the better valid, the smaller is the body. One can imagine it as formulated in terms of some normal coordinates in an ε -order size neighborhood of the point $x \in M$, when ε tends to zero.

The configuration space of an infinitesimal affine body is given by the principal fibre bundle manifold of linear frames FM , or to be more precise, by the connected component of FM , to exclude the nonphysical singular configurations, when the system $e = (e, \dots, e_A, \dots, e)$ fails to be linearly independent. According to the ideas of differential geometry, interpretation of FM as a principal fibre bundle means that the group $GL(n, R)$ acts freely and transitively on this manifold [25],

$$e = (e, \dots, e_A, \dots, e) \rightarrow eL = (\dots, e_B L^B_A, \dots) \quad (11)$$

(do not confuse this action with the left hand side action of $GL(T_x M)$ on $T_x M$ and the manifold of bases in $T_x M$). More precisely, the subgroup of positive-determinant matrices $GL^+(n, \mathbb{R}) \subset GL(n, \mathbb{R})$ acts freely and transitively on the connected component of FM .

Equivalently, instead FM one can use the manifold of co-frames F^*M , i.e., frames of covariant vectors (linear functions on $T_x M$ at all possible $x \in M$). There is an obvious natural diffeomorphism between FM and F^*M given by the duality mapping

$$FM \ni e = (\dots, e_A, \dots) \rightarrow \tilde{e} = (\dots, e^A, \dots), \quad (12)$$

where

$$e^A(e_B) = \langle e^A, e_B \rangle = e^A_i e^i_B = \delta^A_B. \quad (13)$$

Here the group $GL(n, R)$ acts on the left according to the rule:

$$\tilde{e} = (\dots, e^A, \dots) \rightarrow \tilde{e}L = (\dots, L^{-1A}_B e^B, \dots). \quad (14)$$

If $\dim M = n$ then $\dim FM = \dim F^*M = n(n+1)$; this is the number of degrees of freedom.

Any system of local coordinates x^i on M gives rise to the obvious coordinates (x^i, e^i_A) on FM or (x^i, e^i_A) on F^*M ; here e^i_A are components of e_A with respect to x^i . For simplicity we do not distinguish graphically between x^i and their pull-backs $x^i \circ \pi$, $x^i \circ \pi^*$ respectively to FM and F^*M . Here $\pi : FM \rightarrow M$, $\pi^* : F^*M \rightarrow M$ are natural projections which assign to the frames $e_x \in F_x M \subset FM$, $\tilde{e}_x \in F^*_x M \subset F^*M$ their attachment points $x \in M$.

If M is an affine (flat) space with the linear space of translations V , then obviously FM , F^*M are canonically diffeomorphic respectively with $M \times F(V)$, $M \times F(V^*)$; $F(V)$ and $F(V^*)$ denote here the manifolds of frames in V and V^* (more rigorously - their connected components). This is the byproduct of the fact that the tangent and cotangent bundles TM , T^*M are isomorphic respectively with the Cartesian products $M \times V$, $M \times V^*$. Then the degrees of freedom split naturally into translational (M) and internal (V) ones. If M is a curved manifold, this is no longer the case; there is no canonical identification with Cartesian products, and often there is no identification at all because of topological obstacles. But of course, translational motion in M is always well defined due to the projection $\pi : FM \rightarrow M$. If the total motion in FM (always well defined) is given by a curve $\varrho : \mathbb{R} \rightarrow FM$ (\mathbb{R} is the time axis), then the

translational motion is described by the projected curve $\pi \circ \varrho : \mathbb{R} \rightarrow M$. Translational velocity is also well defined and given at the time instant $t \in \mathbb{R}$ by the tangent vector

$$v(t) = (\pi \circ \varrho)'(t) = \left(T\pi \circ \varrho' \right)(t) \in T_{\pi(\varrho(t))}M. \quad (15)$$

Unlike this, neither the internal motion nor internal velocity are well defined in a bare structure- less manifold. If our internal degrees of freedom are constrained to gyroscopic ones, the configuration space becomes (FM, g) , the manifold of g - orthonormal frames. Then the number of degrees of freedom becomes reduced to

$$\dim(FM, g) = \frac{1}{2}n(n+1), \quad (16)$$

i.e., n translational degrees of freedom, parametrized by spatial coordinates x^i and

$$\frac{1}{2}n(n-1) \quad (17)$$

internal gyroscopic degrees of freedom, in a sense like in the flat Euclidean space. In the latter case the configuration space would be identifiable with

$$M \times F(V, g), \quad (18)$$

or rather with its connected components; $F(V, g)$ denotes here the submanifold of g - orthonormal linear frames in V . Some important problem appears here, which is still important when the full system of affine degrees of freedom is concerned. Namely, in virtue of constraints (8) the quantities x^i, e^i_A fail to be functionally independent and they are not generalized coordinates any longer. In a flat space one can help with the problem of defining independent coordinates in the following way: one fixes some particular orthonormal reference frame E in M and then "parametrizes" $F(V, g)$ via (11), using orthogonal matrices $L \in SO(V, g)$. Those in turn are parametrized in any of well- known ways, e.g., using canonical coordinates of the first kind, Euler angles etc. How to follow this pattern in a curved manifold? The only natural way is to introduce some non-holonomic orthonormal reference frame E in M , i.e., some auxiliary field

$$E = (\dots, E_A, \dots) \quad (19)$$

on M , where

$$(E_A | E_B) = g(E_A, E_B) = g_{ij} E^i_A E^j_B = \eta_{AB} \quad (20)$$

and usually we put $\eta_{AB} = \delta_{AB}$, obviously when g (as assumed) is positively definite. If (M, g) is curved (Riemannian), then E must be non-holonomic.

Let us remind that in differential geometry the non-holonomy object Ω of a field of frames E is defined by any of the following equivalent formulas [64]:

$$[E_A, E_B] = \Omega^C_{AB} E_C, \quad dE^A = \frac{1}{2} \Omega^A_{BC} E^B \wedge E^C, \quad (21)$$

where, as usual E^A are elements of the co-frame \tilde{E} dual to E , "d" denotes the exterior differentiation, and for any vector fields X, Y their Lie bracket $[X, Y]$ is analytically expressed by

$$[X, Y]^i = X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}; \quad (22)$$

this vector field is correctly defined, i.e., it does not depend on the choice of coordinates x^i [25]. The object Ω vanishes if and only if E is holonomic, i.e., consist of vector fields tangent to some coordinate lines. The corresponding tensor field on M ,

$$S[E] = \frac{1}{2} \Omega^A{}_{BC} E_A \otimes \tilde{E}^B \otimes \tilde{E}^C, \quad (23)$$

i.e., analytically

$$S^i{}_{jk} = \frac{1}{2} E^i{}_A \left(\frac{\partial}{\partial x^k} E^A{}_j - \frac{\partial}{\partial x^j} E^A{}_k \right), \quad (24)$$

is the torsion of the teleparallelism connection $\Gamma_{\text{tel}}[E]$ built of E ,

$$S^i{}_{jk} = \frac{1}{2} \left(\Gamma_{\text{tel}}[E]^i{}_{jk} - \Gamma_{\text{tel}}[E]^i{}_{kj} \right), \quad (25)$$

The connection $\Gamma[E]$ is analytically given by

$$\Gamma_{\text{tel}}[E]^i{}_{jk} = E^i{}_A \frac{\partial}{\partial x^k} E^A{}_j.$$

Geometrically it is uniquely defined by the demand that the vector fields E_A are all parallel with respect to the corresponding affine connection,

$$\overset{E}{\nabla} E_A = 0, \quad A = 1, \dots, n. \quad (26)$$

When E is non-holonomic, the torsion of $\Gamma[E]$ does not vanish, but the curvature tensor $\mathcal{R}[E]$ of any $\Gamma[E]$ is always vanishing. Incidentally, let us mention that the teleparallelism connection is directly related to the non-holonomic representation of any possible affine connection. Namely, if Γ is some affine connection on M , then its non-holonomic representation with respect to E is given by coefficients $\Gamma^A{}_{BC}$ such that

$$\nabla_C E_B = \Gamma^A{}_{BC} E_A; \quad (27)$$

here ∇ denotes the covariant differentiation in the Γ -sense. One can easily show that

$$\Gamma^i{}_{jk} - \Gamma_{\text{tel}}[E]^i{}_{jk} = E^i{}_A \Gamma^A{}_{BC} E^B{}_j E^C{}_k. \quad (28)$$

In particular,

$$\Gamma_{\text{tel}}[E]^A{}_{BC} = 0; \quad (29)$$

obviously, we mean here non-holonomic coefficients of $\Gamma_{\text{tel}}[E]$ with respect to E itself.

Let us go back to the main problem of our non-holonomic description. Assume that at some time instant $t \in \mathbb{R}$ the structured material point is placed at $x(t) \in M$ and its internal degrees of freedom have the attitude

$$e(t) = (\dots, e_A(t), \dots) \in F_{x(t)}M; \quad e_A(t) \in T_{x(t)}M. \quad (30)$$

The reference frame just passed then is given by

$$E_{x(t)} = (\dots, E_{Ax(t)}, \dots) \in F_{x(t)}M; \quad E_{Ax(t)} \in T_{x(t)}M. \quad (31)$$

Being attached at the same point $x(t) \in M$ the vectors $e_A(t)$ may be expanded with respect to the frame $E_{x(t)}$,

$$e_A(t) = E_{Bx(t)} L^B_A(t). \quad (32)$$

In this way, if E is globally defined and kept fixed, one can interpret the curve

$$\mathbb{R} \ni t \rightarrow L(t) \in \text{GL}(n, \mathbb{R}) \quad (33)$$

as a correct description of the internal part of motion, and the configuration space FM (its connected component) may be identified with the Cartesian product

$$M \times \text{GL}(n, \mathbb{R}) \quad (M \times \text{GL}^+(n, \mathbb{R})). \quad (34)$$

This representation becomes particularly important when gyroscopic constraints are imposed. Then obviously the matrices $L(t)$ become orthogonal (η -orthogonal) and the configuration space is identified with

$$M \times \text{SO}(n, \mathbb{R}) \quad (M \times \text{SO}(\eta)). \quad (35)$$

Such description is very important technically, because, just like in the flat space model, the group $\text{SO}(n, \mathbb{R})$ (in particular $\text{SO}(2, \mathbb{R})$, $\text{SO}(3, \mathbb{R})$) may be parametrized in a variety of standard ways, and in this way some good, non-redundant, generalized coordinates may be constructed.

But even much more: Even without gyroscopic constraints, generalized coordinates (x^i, e^i_A) which are well-defined then, are non-useful in realistic problems. Unlike this, the representation (34) is well-suited to various viable dynamical models. The idea is to use the polar decomposition of L or its two-polar decomposition (singular value decomposition) [48], [49], [50]. In resulting coordinates some physically viable dynamical models are analytically treatable. Because of this the description based on the use of auxiliary non-holonomic frames E is both technically useful and geometrically interesting. It enables one to reduce many curved-space formulae to ones formally very similar to expressions appearing in mechanics of extended bodies in flat spaces [20], [21]. As mentioned, in a bare structureless manifold M there are two kinds of well-defined velocities: generalized velocity of the total motion in FM (or (FM, g)), $\rho'(t)$ parametrized by $(\dots, \frac{dx^i}{dt}, \dots, \frac{d}{dt}e^i_A, \dots)$ and translational motion velocity $v(t) = (\pi \circ \rho)'(t)$,

i.e., (15) parametrized by $\left(\dots, \frac{dx^i}{dt}, \dots\right)$. The quantities $\frac{d}{dt}e^i_A$ fail to be well-defined internal velocities; moreover, they are not components of vectors in M . If some reference frame E is fixed, the time derivatives of L -matrices with components

$$\mathcal{V}(\text{rl})^A_B(t) := \frac{d}{dt}L^A_B(t) \quad (36)$$

are well-defined measures of internal velocity. Usually it is more convenient to use Lie-algebraic objects denoted by Ω_{rl} , $\widehat{\Omega}_{\text{rl}}$ and given by formulas:

$$\Omega_{\text{rl}}^A_B = \left(\frac{d}{dt}L^A_C\right)L^{-1C}_B, \quad (37)$$

$$\widehat{\Omega}_{\text{rl}}^A_B = L^{-1A}_C \frac{d}{dt}L^C_B = L^{-1A}_C \Omega_{\text{rl}}^C_D L^D_B. \quad (38)$$

Obviously, they are non-holonomic velocities, i.e., they are not time derivatives of any generalized coordinates. The label (rl) means relative; they describe internal motion with respect to the fixed reference frame E . Because of this essential dependence on the calculation-helping auxiliary quantity they are perhaps not very convincing, both physically and geometrically. Indeed E does not describe any real geometry. And it would be rather difficult to construct convincing models of kinetic energy (Riemannian geometry of the configuration space) using just these quantities and basing on the analogy with motion of extended bodies in flat space. Much more natural measure of internal motion velocities is based on covariant derivatives, because the affine connection Γ^i_{jk} is a "real" geometry of M , not an auxiliary analytical tool. Usually, although not necessarily, Γ^i_{jk} is the Levi-Civita connection $\{\Gamma^i_{jk}\}$ built of the metric tensor g_{ij} on M :

$$\Gamma^i_{jk} = \{\Gamma^i_{jk}\} = \frac{1}{2}g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m}). \quad (39)$$

Let us remind that $\{\Gamma^i_{jk}\}$ is uniquely defined by the condition that the metric tensor is parallel, i.e., that its covariant derivative with respect to $\{\Gamma^i_{jk}\}$ vanishes,

$$\nabla_k g_{ij} = 0, \quad (40)$$

and, in addition, it is symmetric, i.e., torsion-free:

$$\Gamma^i_{jk} = \Gamma^i_{kj}. \quad (41)$$

In certain problems it may be useful to admit a weaker relationship between affine connection and metric structures, e.g., Riemann-Cartan space, Weyl space, Riemann-Cartan-Weyl space, or no relationship at all. There are also interesting models when only affine connection but no metric structure is assumed in M .

Affine connection enables one to define vectors of internal velocities V^i_A :

$$V^i_A = \frac{D}{Dt}e^i_A = \frac{d}{dt}e^i_A + \Gamma^i_{jk}(x(t))e^j_A(t)\frac{dx^k}{dt}. \quad (42)$$

This is simply the covariant differentiation of the attached vectors e_a along the curve of translational motion in M ; the latter is described by the time dependence of spatial coordinates $x^i(t)$. When (M, Γ) is a flat affine manifold (vanishing curvature and torsion tensors), then obviously, FM trivializes to $M \times F(V)$ and the quantities $\frac{D}{Dt}e^i_A$ reduce in affine coordinates to usual derivatives $\frac{d}{dt}e^i_A$. In a general manifold with non-flat affine connection, the above quantities V^i_A (42) are non-holonomic velocities, i.e., they are not time derivatives of generalized coordinates. And the total covariant velocity $(\dots, v^i, \dots; \dots, V^i_A, \dots)$ with components

$$(\dots, v^i, \dots; \dots, V^i_A, \dots) = \left(\dots, \frac{dx^i}{dt}, \dots; \dots, \frac{De^i_A}{Dt}, \dots \right) \quad (43)$$

is non holonomic in this sense. Let us now express these velocities in terms of the description based on the use of auxiliary (usually g-orthonormal) reference field of frames E .

For any virtual motion, not necessarily one satisfying any equations of motion we have in virtue of (32)

$$V_A = \frac{D}{Dt}e_A = \left(\frac{D}{Dt}E_{x(t)B} \right) L^B_A(t) + E_{x(t)B} \frac{D}{Dt}L^B_A(t). \quad (44)$$

Obviously, from the point view of geometry of M , $L^B_A(t)$ are scalar quantities, thus their covariant derivatives reduce to the usual ones,

$$\frac{D}{Dt}L^B_A(t) = \frac{d}{dt}L^B_A(t). \quad (45)$$

The projected curve $\mathbb{R} \ni t \rightarrow x(t)$ describes translational motion in M (analytically, $x^i(t)$), and the first factor in the first term of (44) will be expressed by the Γ -covariant derivative of the field E :

$$\frac{D}{Dt}E_{x(t)B} = (\nabla_i E_B)_{x(t)} \frac{dx^i}{dt} = (\nabla_C E_B)_{x(t)} E^C_{ix(t)} \frac{dx^i}{dt} \quad (46)$$

In analogy to the flat-space theory of extended bodies we shall use affine velocities, i.e., what Eringen used to call gyration [17] in this theory of micromorphic continua,

$$\Omega^i_j = \left(\frac{D}{Dt}e^i_A \right) e^A_j, \quad \hat{\Omega}^A_B = e^A_i \frac{D}{Dt}e^i_B. \quad (47)$$

Roughly speaking, this is respectively spatial and co-moving representation of the same quantity,

$$\Omega^i_j = e^i_A \hat{\Omega}^A_B e^B_j. \quad (48)$$

If motion is g -rigid, i.e., the frames e_A are constrained to be orthonormal, these tensors are skew-symmetric with respect to the corresponding spatial and material metrics,

$$\Omega^i_j = -\Omega_j^i = -g_{jk}g^{il}\Omega^k_l \quad (49)$$

$$\hat{\Omega}^A_B = -\hat{\Omega}_B^A = -\eta_{BC}\eta^{AD}\hat{\Omega}^C_D \quad (50)$$

and become the usual angular velocities. In exceptional but physical dimension $n = 3$ they are identified (as all skew-symmetric tensors) with axial vectors referred to as angular velocity vectors.

We shall use the following notation for objects appearing in formulas (44-48):

$$\widehat{\Omega}_{\text{rl}}^A{}_B = L^{-1A}{}_C \frac{d}{dt} L^C{}_B, \quad (51)$$

$$\widehat{\Omega}_{\text{dr}}^A{}_B = L^{-1A}{}_F \Gamma^F{}_{DC} L^D{}_B L^C{}_G v^G, \quad (52)$$

where v^G are co-moving components of translational velocity vector,

$$v^G = e^G{}_i \frac{dx^i}{dt} \quad (53)$$

The meaning of labels "rl", "dr" is respectively "relative" and "drive". $\widehat{\Omega}_{\text{rl}}$ describes the affine velocity with respect to the just passed reference frame $E_{x(t)} \in F_{x(t)}M$. $\widehat{\Omega}_{\text{dr}}$ is the affine velocity (angular velocity and strain rate) of the frame E itself as seen by the moving observer. This quantity describes the contribution of translational velocity to the total $\widehat{\Omega}^A{}_B$. This method of non-holonomic frames was intensively used by M. Zórawski in his book and papers on defect theory; [64] and references therein.

Canonical momenta conjugate to generalized coordinates x^i , $\varphi^i{}_A$ will be denoted by p_i , $p^A{}_i$. In many problems it is convenient to use non holonomic momenta conjugate to non-holonomic velocities $\Omega^i{}_j$, $\widehat{\Omega}^A{}_B$, \widehat{v}^A . By analogy to mechanics of extended bodies in flat spaces we shall use the terms affine spin (hypermomentum) respectively in laboratory and co-moving representation, denoted respectively by $\Sigma^i{}_j$, $\widehat{\Sigma}^A{}_B$. As mentioned, the covariant velocities of the total motion are given by

$$(\dots, v^i, \dots; \dots, V^i{}_A, \dots) = \left(\dots, \frac{dx^i}{dt}, \dots; \dots, \frac{D}{Dt} e^i{}_A, \dots \right); \quad (54)$$

if Γ is non-Euclidean, they are non-holonomic,

$$V^i{}_A = v^i{}_A + \Gamma^i{}_{jk} e^j{}_A v^k, \quad (55)$$

$$v^i{}_A = \frac{d}{dt} e^i{}_A. \quad (56)$$

Their conjugate non-holonomic momenta $(p_i, P^A{}_i)$ satisfy:

$$P_i = p_i - e^j{}_A P^A{}_k \Gamma^k{}_{ji}, \quad P^A{}_i = p^A{}_i, \quad (57)$$

and the spin quantities are given by

$$\Sigma^i{}_j = e^i{}_A P^A{}_j, \quad \widehat{\Sigma}^A{}_B = P^A{}_i e^i{}_B = e^i{}_A \widehat{\Sigma}^A{}_B e^B{}_j \quad (58)$$

Let us observe the characteristic duality: v^i is a well-defined vector in M , whereas $v^i{}_A = \frac{d}{dt} e^i{}_A$ fail to be so. Unlike this $P^A{}_i = p^A{}_i$ are components

of well defined co-vectors in M , whereas p_i themselves do not represent co-vector components in M . The all dualities mentioned here and underlying the definitions of p_i , P^A_i , Σ^i_j , $\widehat{\Sigma}^A_B$ are meant in the sense:

$$p_i v^i + P^A_i V^i_A = P_i V^i + P^A_i V^i_A = p_i v^i + \Sigma^i_j \Omega^j_i = \widehat{P}_A \widehat{V}^A + \widehat{\Sigma}^A_B \widehat{\Omega}^B_A \quad (59)$$

where

$$\widehat{P}_A = P_i e^i_A. \quad (60)$$

After some calculations one can show that the usual Poisson brackets for the phase space coordinates have the form

$$\{x^i, x^j\} = \{e^i_A, e^j_B\} = 0, \quad \{x^i, e^j_A\} = \{P^A_i, P^B_j\} = 0, \quad (61)$$

$$\{P^A_i, x^j\} = 0, \quad \{e^i_A, P^B_j\} = \delta^i_j \delta^B_A, \quad \{x^i, P_j\} = \delta^i_j. \quad (62)$$

There is nothing surprising in those brackets; they look exactly as ones with P_i replaced by p_i ; (remember that $P^A_i = p^A_i$; cf. (57)). However, (61) (62) is not a complete system and the Poisson brackets like $\{P_i, P_j\}$, $\{P_i, P^A_j\}$, $\{P_i, e^j_A\}$ are different than their holonomic analogues, like, e.g.,

$$\{p_i, p_j\} = 0, \quad \{p_i, p^A_j\} = 0, \quad \{p_i, e^j_A\} = 0, \quad (63)$$

etc. Indeed, using the expressions (58), one can easily show that

$$\begin{aligned} \{P_i, P_j\} &= \Sigma^k_l R^l_{kij}, \\ \{P_i, P^A_j\} &= -P^A_k \Gamma^k_{ji}, \\ \{P_i, e^j_A\} &= e^k_A \Gamma^j_{ki}, \end{aligned} \quad (64)$$

where R^l_{kij} is the curvature tensor in the convention

$$R^l_{kij} = \Gamma^l_{kj,i} - \Gamma^l_{ki,j} + \Gamma^l_{ai} \Gamma^a_{kj} - \Gamma^l_{aj} \Gamma^a_{ki} \quad (65)$$

and comma denotes the partial derivative with respect to the indicated coordinates x^i . Obviously, in a flat affine space, (64) reduce to (63). The Poisson brackets (64) mean geometrically that, P_i are Hamiltonian generators of the parallel transport of state variables. In a flat space they are just Hamiltonian generators of spatial translations. It is easy to find another Poisson brackets, also convenient in applications, and at the same time admitting a clear geometric interpretation. For example,

$$\{\Sigma^a_b, \Sigma^i_j\} = \delta^a_j \Sigma^i_b - \delta^i_b \Sigma^a_j. \quad (66)$$

Here we easily discover just the well-known structure constants of the linear group. And no wonder, Σ^k_l , which are referred to as affine spin (hypermomentum) components, are Hamiltonian generators of linear transformations of $\text{GL}(T_x M)$ in $FT_x M$ (in the manifold of linear frames, i.e., internal degrees of freedom [1], [2], [8], [30], [48], [49]). Let us remind, if we have the field of non-singular mixed tensors in M ,

$$M \in x \rightarrow L_x \in \text{GL}(T_x M) \subset L(T_x M) \simeq T^1_1(T_x M), \quad (67)$$

it acts on our configuration space, more exactly on the manifold of internal degrees of freedom, according to the rule

$$F_x M \ni e = (\dots, e_A, \dots) \rightarrow (\dots, L_x \circ e_A, \dots), \quad (68)$$

i.e., analytically:

$$(\dots, x^a, \dots; \dots, e^i_A, \dots) \rightarrow (\dots, x^a, \dots; \dots, L^i_j(x) e^j_A, \dots). \quad (69)$$

Obviously, the functions L^i_j above are components of the field L with respect to local coordinates x^i ,

$$L = L^i_j(x) \frac{\partial}{\partial x^i} \otimes dx^j. \quad (70)$$

Obviously, this action is essentially local in the sense of x -dependence of L^i_j on x^k . In general there is no coordinate system in which they would be constant. And if accidentally they happen to be so in some coordinates, this is an artefact; in other coordinates they will depend on x^k (unless L^i_j has the form $\lambda \delta^i_j$, where λ is constant).

The quantities of affine spin, i.e., Σ^a_b are Hamiltonian generators of (68) in action on internal degrees of freedom. So if $L_x = \exp(\alpha(x))$ α being a mixed tensor field on M then (68), (69) become

$$F_x M \ni e = (\dots, e_A, \dots) \rightarrow (\dots, \exp(\alpha(x)) e_A, \dots), \quad (71)$$

i.e., analytically,

$$(\dots, x^a, \dots; \dots, e^i_A, \dots) \rightarrow (\dots, x^a, \dots; \dots, (\exp(\alpha(x)))^i_j(x) e^j_A, \dots). \quad (72)$$

This means that the phase-space functions F depending only on coordinates but not on their conjugate momenta, i.e., $F(x, e)$ suffer the transformation rule

$$F \rightarrow (\mathfrak{L}[\alpha]) F, \quad (73)$$

where the differential operator $\mathfrak{L}[\alpha]$ has the following form in action on such functions:

$$\mathfrak{L}[\alpha] F = \{\alpha^i_j(x) \Sigma^j_i, F\} = \alpha^i_j \{\Sigma^j_i, F\}. \quad (74)$$

Another system of Poisson brackets,

$$\{P_i, \Sigma^k_j\} = \Sigma^l_j \Gamma^k_{li} - \Sigma^k_l \Gamma^l_{ji} \quad (75)$$

show that P_i are Hamiltonian generators of parallel transports along coordinate lines in M , just like Σ^i_j (as we have just seen) are basic Hamilton generators of the system of groups $\text{GL}(T_x M)$. One can easily show that (and this information is already contained in the above ones) for any function F depending only on the configuration, i.e., on the FM -variables (but not on P_i and P^A_i) the following holds:

$$\{\Sigma^i_j, F\} = -E^i_j F = -e^i_A \frac{\partial}{\partial e^j_A} F, \quad (76)$$

where

$$E^i_j = e^i_K e^L_j E^K_L = e^i_K \frac{\partial}{\partial e^j_K}. \quad (77)$$

For (76), (77) no direct analogue holds for functions depending on canonical momenta (excepting the flat-space motion).

Let us write down the co-moving representation of the above brackets, i.e., Poisson brackets for quantities $\hat{P}_A, \hat{\Sigma}^A_B$:

$$\{\hat{P}_A, \hat{P}_B\} = \hat{\Sigma}^K_L R^L_{KAB} - 2\hat{P}_K S^L_{AB}, \quad (78)$$

$$\{\hat{\Sigma}^A_B, \hat{P}_C\} = -\hat{P}_B \delta^A_C, \quad (79)$$

$$\{\hat{\Sigma}^A_B, \hat{\Sigma}^C_D\} = \delta^C_B \hat{\Sigma}^A_D - \delta^A_D \hat{\Sigma}^C_B, \quad (80)$$

$$\{\Sigma^i_j, \hat{\Sigma}^A_B\} = 0, \quad (81)$$

where the curvature and torsion symbols with capital indices denote the co-moving components of the corresponding tensors, so they depend not only on the spatial coordinates x^k but also on the along-fibre coordinates e^i_A :

$$R^L_{KAB} = e^L_i R^i_{jab} e^j_K e^a_A e^b_B, \quad (82)$$

$$S^K_{AB} = e^K_i S^i_{jm} e^j_A e^m_B. \quad (83)$$

Let us observe both similarities and differences in the structure of the co-moving system brackets and that one based on (64, 66, 75). Namely (78, 79, 80) is very similar to the basic commutation relations for the affine group $GAf(n, \mathbb{R})$. Indeed, on the right-hand sides of (78, 79, 80) we "almost" recognize the structure constants of $GAf(n, \mathbb{R}) \simeq \text{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ (\ltimes denotes the semidirect product of $\text{GL}(n, \mathbb{R})$ and \mathbb{R}^n). The proviso "almost" concerns the first subsystem, i.e., (78). Its right hand sides vanish only if the connection Γ is flat, i.e., if we consider the usual affine-space-problem. Surprisingly enough, on the right-hand side the torsion tensor appears, unlike in (64). This fact is both mechanically and geometrically interesting in itself, however for us it is not particularly interesting, because, as a rule, we shall work in a Riemann manifold, where Γ is the usual Levi-Civita connection, with vanishing torsion. Nevertheless, the Riemann-Cartan space, where the torsion is admitted, is also of some interest, first of all when the linear defects (dislocations) are admitted [64]. But the Poisson brackets (79), (80) do not depend on geometry of M and correspond exactly to the structure constants of $GAf(n, \mathbb{R})$. It is seen that \hat{P}_A are Hamiltonian generators of parallel transports along the A -th "ribs" of A (roughly speaking). And similarly $\hat{\Sigma}^A_B$ are Hamiltonian generators of (11, 14). There are however some structural differences between the role of Σ^i_j and $\hat{\Sigma}^A_B$ in this sense. Namely, one can meaningfully say that L in (11) is constant, it is just the system of scalars in M .

Of course, one can consider "micromaterial transformations" where L is local like in gauge theories, i.e., depends on coordinates x^k . But nevertheless there exists a finite - dimensional subgroup, isomorphic with $GL(n, \mathbb{R})$ itself and given by constant matrix elements L^A_B . Unlike this, as mentioned, the constancy of L^i_j in (69, 70) is ill-defined. For the general, x -dependent matrices $[L^A_B] = [(\exp \ell)^A_B]$ we have, in a full analogy to (71, 72)

$$F_x M \ni e = (\dots, e_A, \dots) \rightarrow (\dots, e_B L(x)^B_A, \dots), \quad (84)$$

$$(\dots, x^a, \dots; \dots, e^i_A(x), \dots) \rightarrow (\dots, x^a, \dots; \dots, e^i_B(\exp \ell)^B_A, \dots). \quad (85)$$

In analogy to (73), any function of generalized coordinates $F(x, e)$ suffers the transformation rule

$$F \rightarrow (\exp \Re[\ell]) F, \quad (86)$$

where again the differential operator $\Re[\ell]$ acts on F as follows:

$$\Re[\ell] F = \{\ell^B_A(x) \widehat{\Sigma}^A_B, F\} = \ell^B_A(x) \{\widehat{\Sigma}^A_B, F\}. \quad (87)$$

And one can easily show that for any function F depending only on coordinates x^i, e^i_A we have

$$\{\widehat{\Sigma}^A_B, F\} = -E^A_B F = -e^i_B \frac{\partial F}{\partial e^i_A}, \quad (88)$$

$$\{\widehat{P}_A, F\} = -H_A F, \quad \{P_i, F\} = H_i F, \quad (89)$$

where [30]

$$H_i = e^A_i H_A = \frac{\partial}{\partial x^i} - \Gamma^k_{ji} e^j_B \frac{\partial}{\partial e^k_B}. \quad (90)$$

It is seen again that P_i are Hamiltonian generators of parallel transports along the axis of i -th coordinates x^i and \widehat{P}_A are generators of parallel transports along the A -th legs of linear frames e .

Let us remind the geometric meaning of the above quantities. In geometrical formulation of the theory of affine connection the vector fields E^K_L are so-called fundamental fields of connection acting vertically along fibres in FM . Similarly, H_L are basic horizontal vector fields. Their structure; let us repeat it [30]:

$$E^K_L = e^i_L \frac{\partial}{\partial e^i_K}, \quad H_L = e^i_L \left(\frac{\partial}{\partial x^i} - \Gamma^k_{ij} e^j_A \frac{\partial}{\partial e^k_A} \right) \quad (91)$$

implies that their dual system of covector fields, i.e., differential one-forms ω^K_L, θ^K is given by:

$$\omega^K_L = e^K_i (de^i_L + \Gamma^i_{jk} e^j_L dx^k), \quad \theta^K = e^K_i dx^i. \quad (92)$$

As usual, by duality we mean the Kronecker- delta structure of mutual contractions of basic covariant covectors fields with basic vectors, so:

$$\langle \omega^K_L, E^A_B \rangle = \delta^K_B \delta^A_L, \quad \langle \omega^K_L, H_A \rangle = 0, \quad (93)$$

$$\langle \theta^K, E^A{}_B \rangle = 0, \quad \langle \theta^K, H_A \rangle = \delta^K{}_A. \quad (94)$$

Let us observe the characteristic geometric duality. In (91) $E^K{}_L$ do not depend on the affine connection Γ . They have to do only with the action of $\text{GL}(n, \mathbb{R})$ along fibres (they manipulate with internal degrees of freedom). Unlike this, by the very definition of being horizontal, H_K depend explicitly on the affine connection. And quite dually $[\omega^K{}_L]$ defining horizontal fields as its kernel, is explicitly built of Γ . But θ^K are also Γ -independent; they have to do only with the bundle structure of internal degrees of freedom. Incidentally θ^K do not depend on the choice of coordinates, although apparently they are defined with the use of coordinates x^i , but they, as a matter of fact, do not depend on the choice of x^i . They may be defined without any use of M -coordinates at all, but here there is no place for discussing such purely geometric facts [30]. Let us observe that the above objects (91), (92) in spite of their seemingly abstract geometric nature are nicely interpretable in purely mechanical terms. Namely, let $\varrho : \mathbb{R} \rightarrow FM$ be a curve describing the total motion of our object and let $\dot{\varrho}(t) \in T_{\varrho(t)}FM$ be its generalized velocity at the same time instant $t \in \mathbb{R}$. One can easily show that evaluating the differential forms $\omega^K{}_L$, θ^K on $\dot{\varrho}$, i.e., performing the total contractions of covectors $\omega^K{}_L$, θ^K with the vector $\dot{\varrho}(t)$, one obtains just the object of affine and translational velocities like (49), (50), (53). Indeed, let us remind that the components of (43) represent vectors in FM but not in M (except the components of translational velocity). But evaluating $\omega^K{}_L$, θ^K , we obtain well-defined co-moving components of the affine and translational velocity. Indeed,

$$\langle \omega^K{}_L, \dot{\varrho} \rangle = \widehat{\Omega}^K{}_L, \quad \langle \theta^K, \dot{\varrho} \rangle = \widehat{v}^K. \quad (95)$$

Similarly, let us introduce instead (91, 92) the corresponding expressions:

$$\begin{aligned} E^i{}_j &= e^i{}_K E^K{}_L e^L{}_i = e^i{}_A \frac{\partial}{\partial e^j{}_A}, \\ H_i &= e^M{}_i H_M = \frac{\partial}{\partial x^i} - \Gamma^k{}_{ji} e^j{}_A \frac{\partial}{\partial e^k{}_A}, \\ \omega^i{}_j &= e^i{}_A \omega^A{}_B e^B{}_j = e^K{}_j (de^i{}_K + \Gamma^i{}_{jm} e^j{}_K dx^m), \\ \theta^i &= e^i{}_K \theta^K = dx^i. \end{aligned} \quad (96)$$

This is also system of dual bases (fields of frames in FM),

$$\begin{aligned} \langle \omega^i{}_j, E^k{}_l \rangle &= \delta^i{}_l \delta^k{}_j, \quad \langle \omega^i{}_j, H_k \rangle = 0, \\ \langle \theta^i, E^k{}_l \rangle &= 0, \quad \langle \theta^i, H_k \rangle = \delta^i{}_k \end{aligned} \quad (97)$$

The difference in comparison with (91, 92) is that (97) are assigned to the particular choice of coordinates in M and the corresponding induced coordinates in FM , whereas (91, 92) are "objective", i.e., coordinate-independent. But nevertheless, in analogy to (95),

$$\langle \omega^i{}_j, \dot{\varrho} \rangle = \Omega^i{}_j, \quad \langle \theta^i, \dot{\varrho} \rangle = v^i = \frac{dx^i}{dt}. \quad (98)$$

Roughly speaking (this is a "joke"), one can write:

$$\begin{aligned}\Omega^i_j &= \frac{d\omega^i_j}{dt}, & v^i &= \frac{d\theta^i}{dt}, \\ \widehat{d\Omega}^A_B &= \frac{\omega^A_B}{dt}, & \widehat{v}^A &= \frac{d\theta^A}{dt}.\end{aligned}$$

The doubled g -skewsymmetric part of Σ^i_j and η -skewsymmetric part of $\widehat{\Sigma}^A_B$ are the usual, i.e., metrical, spin quantities. They are given, by analogy to angular velocities, (49, 50) by expressions:

$$S^i_j := \Sigma^i_j - \Sigma_j^i = \Sigma^i_j - g^{ik}g_{jl}\Sigma^l_k, \quad (99)$$

$$\widehat{V}^A_B := \widehat{\Sigma}^A_B - \widehat{\Sigma}_B^A = \widehat{\Sigma}^A_B - \eta^{AC}\eta_{BD}\widehat{\Sigma}^D_C. \quad (100)$$

More precisely, S^i_j is the usual spin and \widehat{V}^A_B is what for objects in a flat space was called by Dyson "vorticity". S^i_j and \widehat{V}^A_B are respectively Hamiltonian generators of left acting (spacial) and right-acting (micromaterial) rotations preserving respectively $g(x)$ and η . Obviously, we mean the action on internal degrees of freedom, and by rotations we meant that in (68) L_x is g_x -orthogonal, $L_x \in O(T_x M, g_x)$, and in (11, 14, 84) L is η -orthogonal, $L \in O(\mathbb{R}^n, \eta)$, i.e., analytically,

$$g_{xab}L_x^a{}_i L_x^b{}_j = g_{xij}, \quad \eta_{AB}L^A{}_K L^B{}_M = \eta_{KM}. \quad (101)$$

It is important to stress that unlike in (58), for deformative motion, \widehat{V}^A_B are not co-moving components of S^i_j :

$$S^i_j \neq e^i{}_A \widehat{V}^A{}_B e^B{}_j; \quad (102)$$

obviously, the same negative statement is true for extended bodies in a flat affine space [48], [49].

Let us stress a very important point: In flat affine spaces there exist concepts like the total and orbital affine momentum (hypermomentum) and total affine momentum with respect to some fixed spatial point. The same is true for the usual (metrical) spin or angular momentum. If x^i are coordinates of the radius-vector of the centre of mass position $x \in M$ with respect to the mentioned fixed origin $\mathfrak{o} \in M$,

$$\overrightarrow{\mathfrak{o}x} = x^i e_i \quad (103)$$

(e_1, \dots, e_n are basic vectors in the translation space V of M), then the total affine momentum with respect to \mathfrak{o} is analytically given by

$$J^i_j = x^i P_j + \Sigma^i_j, \quad (104)$$

where obviously, the first term is the "orbital" affine momentum. Similarly, for the total angular momentum, i.e., the doubled skew-symmetric part of J^i_j , we have the splitting into the "orbital" angular momentum L^i_j and spin S^i_j , i.e.,

angular momentum of the body with respect to the instantaneous position of the centre of mass in M :

$$\mathfrak{J}^i_j = L^i_j + S^i_j. \quad (105)$$

As usual, the doubled skew-symmetric parts are meant in the convention:

$$\begin{aligned} \mathfrak{J}^i_j &= J^i_j - J_j^i = J^i_j - g^{ik} g_{jl} J^l_k, \\ L^i_j &= x^i P_j - x_j P^i = x^i P_j - g^{ik} g_{jl} x^l P_k, \\ S^i_j &= \Sigma^i_j - \Sigma_j^i = \Sigma^i_j - g^{ik} g_{jl} \Sigma^l_k. \end{aligned} \quad (106)$$

In general, only the total quantities are well-balanced, or in special symmetric cases - just conserved. Nothing like the splittings (104, 105) does exist in general curved manifolds. Moreover, it is only internal quantities, i.e., ones related to internal degrees of freedom that is well-defined. Neither orbital nor total quantities do exist at all. The reason is that in a general curved manifold there is no well-defined concept of the radius-vector.

Obviously, Green and Cauchy deformation tensor, $G[e] \in \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$, $C[e] \in T_x^* M \otimes T_x^* M$ (where $e \in F_x M$) are given by the usual formulas, thus, analytically:

$$G[e]_{AB} = g_{ij}(x) e^i_A e^j_B, \quad C[e]_{ij} = \eta_{AB} e^A_i e^B_j. \quad (107)$$

These are standard formulas well-known from the theory of extended affine bodies in a flat space; just the restriction of the general definition to homogeneous deformations. However, in both cases there is some delicate point the overlooking of which may lead to serious mistakes. The trap is hidden in the too automatic use of Schouten tensor notation. Namely, in certain formulas one uses contravariant tensors obtained by the metrical shift of indices, and following the Schouten formalism one writes them simply as:

$$G[e]^{AB} = \eta^{AC} \eta^{BD} G[e]_{CD}, \quad C[e]^{ij} = g^{ik} g^{jl} C[e]_{kl}, \quad (108)$$

According to the usual convention, when some metrics η , g are fixed, the kernel symbols of tensors with η - and g -shifted indices are identical with the primary ones and it is only the level of the indices that encodes the kind of tensor one is dealing with. The same convention is used for the inverse metric tensors, so, to avoid the crowd of characters one used the symbols

$$\eta_{AB}, \quad \eta^{AB}, \quad g_{ij}, \quad g^{ij} \quad (109)$$

respectively for the covariant metric tensors and their contravariant reciprocals. The point is however that in certain formulas one uses contravariant tensors reciprocal to those with coordinates $G[e]_{AB}$, $C[e]_{ij}$. And denoting their components by $G[e]^{AB}$, $C[e]^{ij}$ following (109) would be completely misleading because of the confusion with (108). Therefore, for such an object one has to introduce a new kernel symbol, e.g., $\widetilde{G}[e]$, $\widetilde{C}[e]$ where the following coordinate-independent conditions are satisfied:

$$\widetilde{G}[e]^{AC} G[e]_{CB} = \delta^A_B, \quad \widetilde{C}[e]^{ik} C[e]_{kj} = \delta^i_j. \quad (110)$$

Those conditions are independent of η, g , and except some special values of e , the inequalities hold:

$$\widetilde{G}[e]^{AB} \neq G[e]^{AB}, \quad \widetilde{C}[e]^{ij} \neq C[e]^{ij}. \quad (111)$$

This is one of exceptional situations when the crowd of characters is unavoidable. When postulating and discussing dynamical models one must use transformation properties of all the above-introduced quantities under mappings (68), (69) and (84), (11). It is easy to see that $G[e]$ transforms under (84), (11) as follows:

$$G[eL(x)]_{AB} = G[e]_{CD} L^C{}_A(x) L^D{}_B(x), \quad e \in F_x M. \quad (112)$$

If for any $x \in M$, $L(x)$ is an orthogonal matrix, $L(x) \in O(\mathbb{R}^n, \eta) \subset \text{GL}(n, \mathbb{R})$ then C is invariant:

$$C[eL(x)] = C[e], \quad e \in F_x M, \quad (113)$$

however, there is no particular well-defined rule when $L(x)$ is a general element of $\text{GL}(n, \mathbb{R})$.

Transformation properties under (68), (69) are dual to the above ones. So for any mixed tensor field T on M given locally by $T^i{}_j(x)$ ($T(x) \in \text{GL}(T_x M)$), we have

$$C[T(x)e]_{ij} = C[e]_{kl} T(x)^k{}_i T(x)^l{}_j, \quad e \in F_x M. \quad (114)$$

If for any $x \in M$, $T(x)$ is an isometry in the $g(x)$ -sense, $T(x) \in O(T_x M, g(X))$ then $G[e]$ is invariant:

$$G[T(x)e] = G[e], \quad e \in F_x M, \quad (115)$$

and there is no well-defined rule expressing $G[T(x)e]$ through $G[e]$ alone if $T(x)$ is not an isometry. Those are exactly the familiar rules well-known from mechanics of extended affine bodies in a flat space. Similarly, deformation invariants for internal degrees of freedom are given by the classical formulas. So, for example, we introduce the mixed Green tensors,

$$\widehat{G}[e]^A{}_B := \eta^{AC} G[e]_{CB} \quad (116)$$

and one of possible choice of basic invariants is the following

$$\mathfrak{K}_a[e] := \text{Tr} \left(\widehat{G}[e]^a \right), \quad a = 1, \dots, n. \quad (117)$$

One can easily show that

$$\mathfrak{K}_a[e] := \text{Tr} \left(\widehat{C}[e]^{-a} \right), \quad a = 1, \dots, n, \quad (118)$$

where

$$\widehat{C}[e]^i{}_j := g^{ik} C[e]_{kj}. \quad (119)$$

Due to the Cayley-Hamilton theorem, quantities $\mathfrak{K}_a[e]$ constructed according to the rule (117) but with other values of $a \in \mathbb{Z}$ may be expressed as functions of the above ones.

In spite of the above complete analogy with mechanics of extended affine bodies in flat space, it must be stressed, however that in transformation rules of kinematical quantities some essential changes appear in comparison with the flat space theory. And this is again a kind of "trap". Indeed, careful calculations show that (68, 69) affect velocities as follows:

$$\begin{aligned} {}'V^i &= V^i, \\ {}'\Omega^i_j &= T^i_l \Omega^l_m T^{-1m}_j + V^k (\nabla_k T^i_m) T^{-1m}_j \\ &= T^i_l \Omega^l_m T^{-1m}_j + (\nabla_V T^i_m) T^{-1m}_j, \\ {}'\hat{\Omega}^A_B &= \hat{\Omega}^A_B + e^A_l T^{-1l}_i (\nabla_V T^i_j) e^j_B, \end{aligned} \quad (120)$$

where the "primed" symbols denote the T -transformed quantities and ∇_V is the directional covariant derivative along the translational velocity V . This rule becomes identical with that for flat space only if T is covariantly constant, i.e.,

$$\nabla_k T^i_m = 0. \quad (121)$$

The dual canonical momenta transform as follows:

$$\begin{aligned} {}'P_i &= P_i - \Sigma^k_l T^{-1l}_j \nabla_i T^j_k, \\ {}'\Sigma^i_j &= T^i_k \Sigma^k_m T^{-1m}_j, \\ {}'\hat{\Sigma}^A_B &= \hat{\Sigma}^A_B. \end{aligned} \quad (122)$$

It is seen that the rule for internal quantities (spatial and co-moving components of the affine spin) is identical with that for flat spaces. Unlike this, the translational covariant momentum suffers an additive correction linear in ∇T . The micromaterial local transformations (84) act as follows:

$$\begin{aligned} {}'V^i &= V^i, \\ {}'\Omega^i_j &= \Omega^i_j + e^i_B \left(L^B_{A,k} L^{-1A}_C \right) e^C_j V^k, \\ {}'\hat{\Omega}^A_B &= L^{-1A}_C \hat{\Omega}^C_D L^D_B + L^{-1A}_C L^C_{B,k} V^k, \end{aligned} \quad (123)$$

$$\begin{aligned} {}'P_i &= P_i - \hat{\Sigma}^A_C L^C_{B,i} L^{-1B}_A, \\ {}'\hat{\Sigma}^A_B &= L^{-1A}_C \hat{\Sigma}^C_D L^D_B, \\ {}'\Sigma^i_j &= \Sigma^i_j, \end{aligned} \quad (124)$$

where, obviously, comma denotes the usual partial differentiation of scalar functions. It is seen again that the affine spin transforms in a "proper" way, i.e., just like in a flat space. Translational covariant momentum and velocity variables transform "properly" when L is constant, i.e., when we deal with transformations (11).

3 Equations of motion

The first step is to derive equations of motion for non-dissipative, Lagrangian-Hamilton systems. Later on one introduces the general models by admitting some auxiliary generalized forces of non-Hamiltonian nature, responsible for non-conservative phenomena. The simplest way is to use the basic Poisson brackets introduced above. Kinetic energy of extended affine bodies in a flat space [48], [49] suggests us, by the simple analogy, to use the following formula

$$T = T_{tr} + T_{int} = \frac{m}{2} g_{ij} v^i v^j + \frac{1}{2} g_{ij} V^i{}_A V^j{}_B J^{AB}, \quad (125)$$

where, let us remind, the positive constant $m > 0$ is the mass of the body (inertia of translational motion), and J^{AB} are components of constant, symmetric ($J^{AB} = J^{BA}$) and positively definite micromaterial inertial tensor (inertia of the rotational and deformative motion). But when dealing with internal degrees of freedom J is a primary quantity, no longer the second-order (quadrupole) momentum of the mass distribution. Although such an interpretation may be admissible from the point of view of the mentioned limit transition with the size of the body, there are situations when such a procedure is essentially inadequate, e.g., when dealing with such objects like gas bubbles in fluids, etc.. They have some inertial properties as elementary observation shows, but it is hardly expected that the internal inertia in such "exotic" situations may be "derivable" from something like

$$J^{AB} = \int a^A a^B d\mu(a), \quad (126)$$

where a^K are Lagrange coordinates and the positive measure μ describes the mass distribution. It may be interesting and instructive to rewrite (125) in some modified forms, e.g.,

$$\begin{aligned} T = T_{tr} + T_{int} &= \frac{m}{2} G_{AB} \hat{v}^A \hat{v}^B + \frac{1}{2} G_{KL} \hat{\Omega}^K{}_A \hat{\Omega}^L{}_B J^{AB} \\ &= \frac{m}{2} g_{ij} v^i v^j + \frac{1}{2} g_{ij} \Omega^i{}_k \Omega^j{}_l J[\varphi]^{kl}, \end{aligned} \quad (127)$$

where $J[\varphi]^{ij}$ are spatial, thus variable, components of the inertial tensor,

$$J[\varphi]^{kl} = \varphi^k{}_A \varphi^l{}_B J^{AB}. \quad (128)$$

After performing the Legendre transformations for Lagrangians of the form $L = T - \mathfrak{U}(x^i, e^j{}_A)$ (no generalized, i.e., velocity-dependent potentials like, e.g., magnetic ones) and expressing generalized velocities by canonical momenta, we obtain the following expression for the kinetic energy (125) :

$$\mathfrak{T} = \mathfrak{T}_{tr} + \mathfrak{T}_{int} = \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} \tilde{J}_{AB} P^A{}_i P^B{}_j g^{ij}, \quad (129)$$

where p_i , P^A_i are canonical momenta conjugate respectively to x^i , e^i_A and \tilde{J} is the covariant inverse of J ,

$$J^{AC} \tilde{J}_{CB} = \delta^A_B. \quad (130)$$

Warning: \tilde{J} must be not confused with the "covariant η -shift of J ", i.e.,

$$\tilde{J}_{AB} \neq \eta_{AC} \eta_{BD} J^{CD}. \quad (131)$$

More precisely, (129) is related to (125) by the following explicit expression for the Legendre transformation:

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = \frac{\partial T}{\partial \dot{x}^i} = m g_{ij} \frac{dx^j}{dt}, \quad (132)$$

$$P^A_i = \frac{\partial L}{\partial \dot{e}^i_A} = \frac{\partial T}{\partial \dot{e}^i_A} = g_{ij} \frac{d\varphi^j_B}{dt} J^{BA}. \quad (133)$$

Energy is given by the usual formula

$$E = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} + \dot{e}^i_A \frac{\partial L}{\partial \dot{e}^i_A} - L = T + \mathfrak{U}(x^i, e^j_B) \quad (134)$$

and after Legendre transformation it becomes Hamiltonian

$$H = \mathfrak{T} + \mathfrak{U}(x^i, e^j_B). \quad (135)$$

In analogy to (127) the kinetic Hamiltonian (129) may be written in the form

$$\begin{aligned} \mathfrak{T} = \mathfrak{T}_{tr} + \mathfrak{T}_{int} &= \frac{1}{2m} \tilde{G}^{AB} \hat{p}_A \hat{p}_B + \frac{1}{2} \tilde{J}_{AB} \hat{\Sigma}^A_C \hat{\Sigma}^B_D \tilde{G}^{CD} \\ &= \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} \widetilde{J[\varphi]}_{kl} \Sigma^k_i \Sigma^l_j g^{ij}. \end{aligned} \quad (136)$$

This expression is built of geometric quantities-Hamiltonian generators of important transformations acting in the configuration and phase spaces. More precisely, it is a quadratic form of generators. Coefficients of this quadratic form depend on the configuration variables x^i , e^i_A , i.e., on the position of the body in M and internal parameters.

The simplest way to derive equations of motion is not to derive them from Lagrange equations. This would be the terrible, very strenuous work where mistakes are simply unavoidable and the structure of resulting equations is non-readable. Rather, at least for non-dissipative systems, the best procedure is to use the above-quoted basic Poisson brackets and to write equations of motion in the form

$$\frac{dF}{dt} = \{F, H\}, \quad (137)$$

where H is the Hamiltonian, e.g., (135, 136) and F runs over some complete systems of functionally independent functions on the phase space (some non-traditional in the sense of Darboux coordination of our phase space). Later

on, one may perform the inverse Legendre transformation and go back to the usual second-order equations of motion equivalent to Lagrange equations of the second kind. For certain reasons it is instructive to begin with a rather academic situation when there is no a priori assumed relationship between affine connection Γ and metric tensor g in our space manifold M . Then as a matter of fact we are given two affine connections: Γ^i_{jk} and Levi-Civita connection $\{^i_{jk}\}$ induced by g . Obviously, as usual, the difference of two affine connections

$$\mathfrak{K}^a_{bc} := \Gamma^a_{bc} - \{^a_{bc}\} \quad (138)$$

is a tensor field (once contravariant, twice covariant) although Γ^a_{bc} , $\{^a_{bc}\}$ separately have not this property.

Performing the above Poisson-bracket-procedure (137) of deriving equations of motion and inverting Legendre transformation, one obtains the following suggestive system of second-order differential equations written in a balance-like form:

$$\begin{aligned} m \frac{Dv^i}{Dt} &= m \mathfrak{K}^i_{jk} v^j v^k + \Sigma^m_n R^n_m{}^i{}_j v^j \\ &\quad - \mathfrak{K}_{mn}{}^i \frac{De^m_K}{Dt} \frac{De^n_L}{Dt} J^{KL} + F^i, \end{aligned} \quad (139)$$

$$e^i_K \frac{D^2 e^j_L}{Dt^2} J^{KL} = -e^i_K g^{jm} \frac{Dg_{mn}}{Dt} \frac{De^n_L}{Dt} J^{KL} + N^{ij}, \quad (140)$$

where the meaning of symbols is as follows:

1. the shift of indices (up-down) is meant in the sense of metric g ,
- 2.

$$N^{ij} = N^i_k g^{kj}, \quad N^i_k = -e^i_A \frac{\partial \mathfrak{U}}{\partial e^k_A} = -E^i_k \mathfrak{U} \quad (141)$$

(cf (77)),

- 3.

$$F^i = g^{ik} F_k, \quad F_k = -H_k \mathfrak{U} = -\frac{\partial \mathfrak{U}}{\partial x^k} + \Gamma^a_{bk} e^b_B \frac{\partial \mathfrak{U}}{\partial e^a_B} \quad (142)$$

(cf (90)).

Let us mention that $H_k \mathfrak{U}$ are components of what is known in mechanics of principal bundles of frames as a covariant exterior differential $D\mathfrak{U}$ of the scalar function \mathfrak{U} . Note that if \mathfrak{U} is not projectable from FM to M (i.e., if it depends not only on x^i but also on e^i_A), then $D\mathfrak{U} \neq d\mathfrak{U}$ unless the affine connection Γ is flat. Let us notice also that in a flat affine space (Euclidean space) our equations of motion (139), (140) reduce to the well-known equations of motion of extended affinely rigid bodies [48], [49]. And also in curved spaces the main ideas of interpretation are similar to the mentioned ones. On the left-hand side of (139) we have covariant acceleration multiplied by mass. Indeed, $v^i = \frac{dx^i}{dt}$ is just the usual velocity of translational motion. On the right-hand side we

have the usual force F^i acting on the material point. Similarly, in (140) N^{ij} is just what was called affine momentum, or hyperforce, influencing directly the motion of internal degrees of freedom. We have started from variational principle, therefore, these dynamical quantities F^i , N^{ij} have the very peculiar potential structure being build of derivatives of \mathfrak{U} . Incidentally, let us observe a very interesting relationship:

$$F_k = -\frac{\partial \mathfrak{U}}{\partial x^k} - N^a{}_b \Gamma^b{}_{ak}. \quad (143)$$

However, once derived in this way, equations of motion (139), (140) may be easily generalized by admitting dissipative phenomena. Simply, we may allow the force F^k and hyperforce N^{ab} to be completed by some completely non-potential terms depending on velocities v^i , $V^i{}_A$ or equivalently v^i , $\Omega^i{}_j$. Such additional terms may describe viscous friction, both in translational and internal motion. In this sense, without the particular model (141-142) equations (139), (140) are quite general.

Let us now concentrate on peculiarities of motion in curved spaces. In a sense the right-hand side of (139) may be interpreted as the total force F^i_{tot} affecting translational motion and responsible for its covariant acceleration. But except the external force F^i , this total force F^i_{tot} contains certain additional terms. In an obvious way those additional terms represent the geometric force in the sense of coupling between geometry of M and the particle degrees of freedom, both internal and translational one. Similarly, in (140) besides the usual "external" term N^{ij} , on the right hand side there is an additional geometric term describing the coupling of internal degrees of freedom with spatial structure. Notice that even if external translational force vanishes, $F^i = 0$, translational motion is not geodesic. This is just the consequence of the mentioned coupling between geometry of M and mechanical degrees of freedom of the particle. If we insisted on approximate description of a small extended body, this phenomenon would have to do with geodesic deviation. Equations of motion (139), (140) simplify in a remarkable way when we go back to the model where affine connection Γ is implied by metrical structure g , or at least when there exists some kind of compatibility between Γ and g .

The most natural model, applicable both in defect theory [64] and in General Relativity is that of Riemann-Cartan space where metric g is assumed to be parallel under affine connection Γ ,

$$\nabla_k g_{ij} = 0. \quad (144)$$

Then $\mathfrak{K}^a{}_{bc}$ becomes the so-called contorsion tensor $K^a{}_{bc}$ and it is easily shown in differential geometry that [64]

$$K^a{}_{bc} = S^a{}_{bc} + S_{bc}{}^a + S_{cb}{}^a \quad (145)$$

where S is the torsion tensor of Γ ,

$$S^i{}_{jk} = \frac{1}{2} (\Gamma^i{}_{jk} - \Gamma^i{}_{kj}). \quad (146)$$

All tensor indices are manipulated in the sense of g ; in particular, contorsion is g -skew-symmetric in the first pair of indices,

$$K^a{}_{bc} = -K_b{}^a{}_c = -g_{bi}g^{aj}K^i{}_{jc}. \quad (147)$$

In this way Γ is controlled by two independent and a priori arbitrary quantities: g and S . Equations of motion (139, 140) simplify then to the following form:

$$m \frac{Dv^i}{Dt} = \Sigma^a{}_b R^b{}_a{}^i{}_j v^j + 2mv^b v^c S_{bc}{}^i + F^i, \quad (148)$$

$$e^i{}_A \frac{D^2}{Dt^2} e^j{}_B J^{AB} = N^{ij}. \quad (149)$$

In Riemann-Cartan spaces the curvature tensor is g -skew-symmetric in the first pair of indices,

$$R^b{}_{aij} = -R_a{}^b{}_{ij} = -g_{ac}g^{bd}R^c{}_{dij}; \quad (150)$$

Because of this (150) g -skewsymmetry it is only the g -skew-symmetric part of Σ that really enters the formula (148). But this skew-symmetric part is just the half of internal angular momentum (spin) in the spatial representation,

$$\frac{1}{2} (\Sigma^i{}_j - \Sigma_j{}^i) = \frac{1}{2} (\Sigma^i{}_j - g^{ik}g_{jm}\Sigma^m{}_k) = \frac{1}{2} S^i{}_j \quad (151)$$

cf. (99). Therefore (148) becomes:

$$\frac{Dp^i}{Dt} = \frac{Dv^i}{Dt} = 2mv^a v^b S_{ab}{}^i + \frac{1}{2} S^a{}_b R^b{}_a{}^i{}_j v^j + F^i \quad (152)$$

and the similar balance-like form of (149) reads

$$\frac{D\Sigma^{ij}}{Dt} = \tilde{J}_{ab}\Sigma^{ai}\Sigma^{bj} + N^{ij}, \quad (153)$$

where the e -dependent inertial tensor \tilde{J}_{ab} is given by

$$\tilde{J}_{ab} = \tilde{J}_{KL} e^K{}_a e^L{}_b, \quad (154)$$

or, equivalently, in reciprocal contravariant terms

$$J^{ab} = e^a{}_K e^b{}_L J^{KL}. \quad (155)$$

Equations of motion (152, 153) are represented here as a system of balance laws for the kinetic linear momentum and affine spin p^i , Σ^{ij} , where according to Legendre transformation

$$p^i = mv^i, \quad \Sigma^i{}_j = g_{ja}\Omega^a{}_b J^{bi}, \quad \Sigma^{ij} = \Omega^j{}_b J^{bi}. \quad (156)$$

Let us stress that unlike in the usual formula

$$P^A{}_i = g_{ij} \frac{De^j{}_B}{Dt} J^{BA}, \quad (157)$$

the "coefficients" J^{bi} in the non-holonomic representation (156) depend on the internal configuration e . So, they are non-constant and are state-dependent even in the flat-space theory. Formally the internal part of dynamics (149), (153) looks like in mechanics of extended affine bodies in Euclidean space. The difference is only that the usual time derivative $\frac{d}{dt}$ is replaced by the covariant one $\frac{D}{Dt}$. The main novelty, as mentioned above, appears in the dynamics of translational motion (148), (152). Because now it is not only so that the "flat" $\frac{dp_i}{dt}$ is replaced by $\frac{Dp_i}{Dt}$. Namely, in addition to the "external" translational force F^i two "geometric" forces appear on the right-hand side of (148), (152). One of them describes the direct coupling between linear momentum (thus also translational velocity) and torsion tensor. The other one describes the direct coupling between internal angular momentum (spin) and the curvature tensor R . Those geometric forces have the "magnetic-like" structure in the sense that they are g -orthogonal to translational velocity, so they do not do any work. The duality: translations-torsion, rotations-curvature is well known in differential geometry [64], and on the mechanical level-in defect theory (dislocations and interstitials or voids) [3], [64]. This is a nice picture, however, let us observe that (148, 152) may be also written in shorter form, where the translation-torsion term apparently disappears,

$$\frac{\delta p^i}{\delta t} = m \frac{\delta v^i}{\delta t} = \frac{1}{2} S^a{}_b R^b{}_{a^i j} v^j + F^i, \quad (158)$$

where $\frac{\delta}{\delta t}$ denotes the covariant differentiation in the g -Levi-Civita sense. Obviously, if there is no torsion then $\frac{\delta}{\delta t} = \frac{D}{Dt}$. The above equations of motion are structurally similar to generally-relativistic equations for the pole-dipole particle. The mentioned generally relativistic models were investigated by Mathisson, Weyssenhof, Papapetrou, Tulczyjew, Künzle and many others [32], [37], [57]. The characteristic couplings: translations-torsion, rotations-curvature have a very deep geometric background and in this way, "geometric forces" on the right-hand side of (148, 152), although derived from some dynamical Lagrangian postulates, are based on some kind of a priori. We concentrated here on the Riemann spaces, when $\nabla g = 0$ and torsion tensor does vanish. Riemann-Cartan space is in a sense more natural, although there appear some ambiguities in the definition of angular velocity and affine velocity when torsion is admitted. We do not deal here with more general relationships between Γ and g , although some of them should be mentioned as interesting in some possible future investigations. Riemann-Cartan-Weyl space is defined as such one in which parallel transport preserves angles between vectors but not necessarily their lengths (roughly speaking parallel transport acts on vectors as a conformal transformation). Then we have

$$\nabla_k g_{ij} = -Q_k g_{ij}, \quad (159)$$

where

$$Q^k = g^{km} Q_m \quad (160)$$

is traditionally referred to as the Weyl vector (Q_k is then the Weyl co-vector). Then (138) has the form

$$\mathfrak{K}^i_{jk} = S^i_{jk} + S_{jk}^i + S^k_{ji} + (\delta^i_j Q_k + \delta^i_k Q_j - g_{jk} Q^i), \quad (161)$$

where, as usual, tensor indices are shifted in the g -sense. If Γ is symmetric (no torsion) then obviously:

$$\mathfrak{K}^i_{jk} = \delta^i_j Q_k + \delta^i_k Q_j - g_{jk} Q^i \quad (162)$$

and one is dealing with what is usually called the Weyl space. Riemann-Cartan-Weyl spaces and Weyl spaces are interesting in themselves and useful in defect dynamics [64]. However, the problem of how to define angular and affine velocity becomes here essential. In any case in this paper we concentrate on Riemann-, or sometimes- Riemann-Cartan spaces. In mechanics of extended affine bodies in flat spaces one often discusses the problem of affine dynamics with additional constraints [48], [49], [60]. The most important example is that of gyroscopic motion, when the body is metrically-rigid. By analogy we can, and in many problems just should, discuss the motion of infinitesimal gyroscope in a curved manifold. According to gyroscopic constraints the frame e should be g -orthonormal during the motion; analytically:

$$\eta_{AB} e^A_i(t) e^B_j(t) = g_{ij}(x(t)). \quad (163)$$

This means that affine velocity Ω^i_j is permanently g -skew-symmetric:

$$\Omega^i_j = -\Omega_j^i = -g_{jk} \Omega^k_m g^{mi}, \quad (164)$$

just angular velocity in spatial representation. The corresponding equations of motion may be obtained from (148-149), i.e., from (152, 153), even without the use of variational principle, just basing on the d’Alambert principle of ideal constraints. The power of generalized forces F^i , N^{ij} on virtual generalized velocities v^i , Ω^i_j is given by

$$\mathfrak{P} = F_i v^i + N^i_j \Omega^j_i. \quad (165)$$

According to the d’Alambert principle, equations of gyroscopically constrained motion are obtained in the following way:

1. on the right-hand side of unconstrained equations one introduces additional reaction forces F_R , N_R which maintain constraints,
2. one substitutes formally constraints equations,
3. one assumes that F_R , N_R do not do any work on virtual displacements compatible with constraints:

$$\mathfrak{P}_R = F_{Ri} v^i + N_{Ri}^j \Omega^j_i = 0 \quad (166)$$

for any v and any Ω satisfying (164).

The result is that F_R does vanish and N_R is g -symmetric, i.e., its g -skew-symmetric part does vanish:

$$F_R^i = 0, \quad N_R^i{}_j - N_{Rj}^i = N_R^i{}_j - g_{jk} N_R^k{}_m g^{ml} = 0 \quad (167)$$

or briefly:

$$F_R^i = 0, \quad N_R^{ij} - N_R^{ji} = 0. \quad (168)$$

This means that the effective reactions-free system of equations of motion consist of (148) (equivalently (152, 158), the skew-symmetric part of (149) (equivalently one of (153)) and constraints equations (163). The mentioned skew-symmetric part of internal equation

$$\left(e^i{}_A \frac{D^2}{Dt^2} e^j{}_B - e^j{}_A \frac{D^2}{Dt^2} e^i{}_B \right) J^{AB} = N^{ij} - N^{ji} \quad (169)$$

has the following balance form given by the skew-symmetric part of (153)

$$\frac{DS^{ij}}{Dt} = N^{ij} - N^{ji} \quad (170)$$

where, obviously, $S^i{}_j$ are spin components (99, 151). The skew-symmetric internal hyperforce

$$\mathfrak{N}^{ij} = N^{ij} - N^{ji} \quad (171)$$

is just the usual torque, i.e., moment of forces acting on the body. In n dimensions it is just a skewsymmetric tensor; the peculiarity of the physical dimension $n=3$ is that it may be identified in a known way with the axial vector \mathfrak{N}^i .

Just like in mechanics of extended affine bodies in flat spaces one can consider also other physically interesting constraints like incompressible motion, rotation-less motion etc. The corresponding equations of motion consist of (152) i.e. (158) and respectively the trace-less part of symmetric part etc. of (149) i.e. (153), and, of course, equations of constraints themselves.

The above balance form of equations of motion is very instructive and reveals geometric foundations of the model, first of all its symmetry properties and the corresponding conservation laws. Another problem is how to solve equations of motion and determine the phase portrait, at least qualitatively. And here the problem of using the non-holonomic reference frame E is crucial. There are two main reasons for that:

1. As already mentioned, when gyroscopic or other constraints are imposed, the quantities $(x^i, e^i{}_A)$, or equivalently $(x^i, e^A{}_i)$ are no longer independent generalized coordinates. Then the best way to introduce reasonable and computationally effective generalized coordinates is just to introduce an auxiliary field of (co)frames E , use representation (32), confine matrices L to be elements of $SO(n, \mathbb{R})$ (gyroscopic model) or some other subgroup $G \subset GL(n, \mathbb{R})$, and parametrize G with the use of some natural coordinates e.g., canonical coordinates of the first or second kind, Euler angles, some byproducts etc.

2. Even if no additional constraints are imposed on affine motion of internal degrees of freedom, it is rather a rule than exception that the natural coordinates (x^i, e^i_A) on FM are inconvenient and non-effective in study of realistic problems like elastic vibrations and their coupling with rigid rotations. For example, one deals often with isotropic problems when the potentials energy \mathfrak{U} is built of deformation invariants. The only reasonable procedure is then to use (32) and express L in terms of left or right polar decomposition or two-polar decomposition (singular value decomposition). Incidentally, this enables one to use description as similar as possible to the one of extended bodies in flat space.

For any $L \in \text{GL}(n, \mathbb{R})$ we have two version of the polar decomposition, sometimes referred to as the left or right one:

$$L = OS = \Sigma O, \quad \Sigma = OSO^{-1}, \quad (172)$$

where $O \in O(n, \mathbb{R})$ is orthogonal and $S = S^T$, $\Sigma = \Sigma^T$ are symmetric and positively-definite. Performing diagonalization of S with the help of some orthogonal matrix $V \in O(n, \mathbb{R})$,

$$S = VDV^{-1}, \quad (173)$$

D being diagonal and positive, and denoting:

$$U = OV \in O(n, \mathbb{R}) \quad (174)$$

we obtain the singular value decomposition (two-polar decomposition):

$$L = UDV^{-1}, \quad U, V \in O(n, \mathbb{R}), \quad D - \text{diagonal}. \quad (175)$$

It is well known that the polar decomposition is unique, but the singular value decomposition suffers some kind of multivaluedness which is essentially harmless if properly treated [48], [49].

In (172) internal degrees of freedom are represented as consisting of two subsystems: rigid body in n -dimensions and deformations; respectively $\frac{1}{2}n(n-1)$ and $\frac{1}{2}n(n+1)$ degrees of freedom. There are two possible representations of deformative modes, as seen in (172). The symmetric-deformative objects have respectively to do with the Green and Cauchy deformation tensors. Namely, matrices of those tensors are given by

$$G = L^T L = S^2, \quad C = \Sigma^{-2}, \quad C^{-1} = \Sigma^2. \quad (176)$$

In the two-polar decomposition L consists of two fictitious rigid bodies represented respectively by U and V ; every with $\frac{1}{2}n(n-1)$ degrees of freedom, and of n purely scalar deformations (stretching) which tell us only how the body is stretched, but without any information about orientations of this stretching both in physical space and in material of the body. The quantities U, V describe

orientations of stretching i.e., material and spatial position of the main axes of Green and Cauchy deformation tensors,

$$G = VD^2V^{-1}, \quad C = UD^2U^{-1}, \quad C^{-1} = UD^{-2}U^{-1}. \quad (177)$$

Having rigid bodies we can introduce their angular velocities, both "spatial" and "co-moving" versions. Obviously the spatial representations are given by :

$$\omega_{rl} = \frac{dO}{dt}O^{-1}, \quad \chi_{rl} = \frac{dU}{dt}U^{-1}, \quad \vartheta_{rl} = \frac{dV}{dt}V^{-1}, \quad (178)$$

respectively for the polar and two-polar decomposition. Similarly, the "co-moving" expressions have the known form:

$$\hat{\omega}_{rl} = O^{-1}\frac{dO}{dt}, \quad \hat{\chi}_{rl} = U^{-1}\frac{dU}{dt}, \quad \hat{\vartheta}_{rl} = V^{-1}\frac{dV}{dt}. \quad (179)$$

Obviously, they are related to each other in the usual way, justifying the terms spatial and co-moving

$$\omega_{rl} = O\hat{\omega}_{rl}O^{-1}, \quad \chi_{rl} = U\hat{\chi}_{rl}U^{-1}, \quad \vartheta_{rl} = V\hat{\vartheta}_{rl}V^{-1}; \quad (180)$$

We must remember however that this is something a bit else than the usual relationship between, e.g., Ω and $\hat{\Omega}$.

The labels "rl" or "dr" at those angular velocities refer to the fact that those quantities describe the L -motion relative with respect to the reference frame E , usually non-holonomic one. It is not only geometrically interesting but also computationally effective to express the kinetic energy of internal motion T_{int} , cf (125) through the above quantities. One obtains:

$$T_{\text{int}} = -\frac{1}{2}\text{Tr}(SJS\hat{\omega}^2) + \text{Tr}\left(SJ\frac{dS}{dt}\hat{\omega}\right) + \frac{1}{2}\text{Tr}\left(J\left(\frac{dS}{dt}\right)^2\right), \quad (181)$$

where

$$\hat{\omega} = \hat{\omega}_{\text{dr}} + \hat{\omega}_{rl} = \hat{\omega}_{\text{dr}} + O^{-1}\frac{dO}{dt}; \quad (182)$$

it is clear that, $\hat{\omega}_{\text{dr}}$ is the restriction of $\hat{\Omega}_{\text{dr}}$ (52) to the rigid motion of the O -gyroscope,

$$\hat{\omega}_{\text{dr}B}^A = O^{-1A}{}_F \Gamma^F{}_{DC} U^D{}_B U^C{}_E \hat{V}^E. \quad (183)$$

Remark: do not confuse completely different things denoted by the same kernel symbol S : torsion $S^i{}_{jk}$, spin $S^i{}_j$, and deformation-symmetric part of the polar decomposition S^{AB} ; unfortunately letters are missing.

In expression (181) we have used the standard orthonormal coordinates in \mathbb{R}^n . Then the micromaterial metric is analytically given by the Kronecker symbol,

$$\eta_{AB} = \delta_{AB}. \quad (184)$$

Obviously, if for any reasons convenient, we can use general rectilinear coordinates in \mathbb{R}^n (no (184)). Then in (181) instead J we must use J_η given by

$$J_\eta^A{}_B = J^{AC} \eta_{CB}. \quad (185)$$

The polar decomposition is convenient when J is general and the internal potential energy \mathfrak{U} is spatially isotropic,

$$\mathfrak{U}(L(x)e(x)) = \mathfrak{U}(e(x)) \quad (186)$$

for any isometry $L(x) \in O(T_x M, g_x)$ at any $x \in M$ acting according to (69–68). In other words, \mathfrak{U} depends on e through the Green deformation tensor G (107). The singular value decomposition (two-polar decomposition (175)) is convenient in doubly isotropic problems, i.e., ones isotropic both in the physical and the micromaterial space. This means two things:

1. Inertial tensor J is invariant under $O(n, \eta)$ acting through (12, 14), therefore,

$$J^{AB} = I \eta^{AB} = {}^* I \delta^{AB}; \quad (187)$$

the last expression based on the natural choice of orthogonal coordinates in \mathbb{R}^n ,

2. Potential energy satisfies both (186) and

$$\mathfrak{U}(e(x)L) = \mathfrak{U}(e(x)) \quad (188)$$

for any $L \in O(n, \mathbb{R})$ acting through (12, 14). This means that \mathfrak{U} depends on e only through deformation invariants (117, 118).

If those conditions are satisfied, the singular value decomposition provides the most effective coordinatization of FM and the formula for kinetic energy becomes:

$$T_{\text{int}} = -\frac{I}{2} \text{Tr} (D^2 \hat{\chi}^2) - \frac{I}{2} \text{Tr} (D^2 \hat{\vartheta}^2) + I \text{Tr} (D \hat{\chi} D \hat{\vartheta}) + \frac{I}{2} \text{Tr} \left(\left(\frac{dD}{dt} \right)^2 \right), \quad (189)$$

where now:

$$\begin{aligned} \hat{\vartheta} &= V^{-1} \frac{dV}{dt}, \\ \hat{\chi} &= \hat{\chi}_{\text{dr}} + \hat{\chi}_{\text{rl}} = \hat{\chi}_{\text{dr}} + U^{-1} \frac{dU}{dt}, \\ \hat{\chi}_{\text{dr}} &= U^{-1A} {}_F \Gamma^F {}_D C L^D {}_B L^C {}_E \hat{V}^E. \end{aligned} \quad (190)$$

There is no drive term in $\hat{\vartheta}$. Expressions (181, 189) have some very peculiar features, interesting from the geometric and analytic point of view, and at the same time very convenient in physical calculations. Namely, formally they are identical with the corresponding formulas for affine motion in flat spaces. The

difference is that $\hat{\chi}$ and $\hat{\omega}$ contain the additional "drive" terms $\hat{\chi}_{\text{dr}}, \hat{\omega}_{\text{dr}}$. These terms depend on geometry of M . Translational velocity occurs in these terms, because of this, they interfere somehow in (125) with the translational term T_{tr} . We shall discuss some special examples in two-dimensional spaces $n = 2$. The peculiarity of dimension two is that the group $O(2, \mathbb{R})$ is Abelian-the exception among the groups $O(n, \mathbb{R})$ because for any $n > 2$ they are semi-simple. Because of this commutativity, the special and co moving representations of angular velocity coincide:

$$\hat{\vartheta} = \vartheta, \quad \hat{\chi}_{\text{rl}} = \chi_{\text{rl}}, \quad \hat{\chi} = \chi.$$

Because of this, another peculiarity appears, namely, there exists an interesting class of integrable models with directly separable Hamilton-Jacobi equations. However, before doing this, for completeness, we quote the co-moving form of balance equations (148, 149, 152, 153). They are curved-space affine counterparts of Euler equations known from rigid body mechanics in Euclidean space. Namely, after some calculations one can obtain:

$$m \frac{dv^A}{dt} = -m \hat{\Omega}^A{}_B v^B + 2v^B v^C S_{CB}{}^A + \frac{1}{2} S^C{}_D R^D{}_C{}^A{}_B v^B + F^A, \quad (191)$$

$$\frac{\hat{\Omega}^B{}_C}{dt} J^{CA} = -m \hat{\Omega}^B{}_D \hat{\Omega}^D{}_C J^{CA} + N^{AB}, \quad (192)$$

where the capitals refer to the co-moving representation, e.g., $v^A = e^A{}_i v^i$, etc., and their raising and lowering is meant in the sense of micromaterial metric η_{AB} (usually δ_{AB} ; rectilinear orthogonal coordinates in \mathbb{R}^n are most convenient). Expressing velocities and affine velocities in terms of linear momentum and spin in co-moving representation, we obtain the following balance laws:

$$\frac{dp^A}{dt} = -p^B \tilde{J}_{BC} \hat{\Sigma}^{CA} + \frac{2}{m} p^B p^C S_{CB}{}^A + \frac{1}{2m} S^D{}_C R^C{}_D{}^A{}_B p^B + F^A \quad (193)$$

$$\frac{\hat{\Sigma}^{AB}}{dt} = -\hat{\Sigma}^{AC} \tilde{J}_{CD} \hat{\Sigma}^{DB} + N^{AB}, \quad (194)$$

The Euler-like structure of those equations is easily readable. Obviously, there is nothing wrong in that on the left- hand side of equations we have the usual differentiation. Though from the point of view of geometry of M , the capital-indices-quantities, i.e., co-moving components, are just scalars. Their covariant derivatives along curves are therefore identical with the usual derivatives.

4 Examples. Two-dimensional homogeneously deformable body.

Let us present as interesting examples the two-dimensional body moving in constant-curvature spaces, i.e., the spherical space $S^2(0, R)$ and pseudo-spherical

Lobachevsky space $H^{2,2,+}(0, R)$. There are realistic situations when the model is rigorously solvable. It is a good illustration of how our method of nonholonomic frames works practically. We also obtain an interesting class of models integrable in the Liouville sense. In "polar" coordinates (r, φ) the corresponding metric elements are given respectively by

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\varphi^2, \quad (195)$$

$$ds^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right) d\varphi^2.$$

In the spherical case all situations with $r = 0$, $r = \pi R$ and arbitrary values of φ correspond to the same points, the "north" pole if $r = 0$ and the "south" pole if $r = 2\pi$. The range of r is $[0, \pi R]$. For the pseudo-spherical Lobachevsky space $H^{2,2,+}(0, R)$ the range of r is $[0, \infty]$. The most convenient choice of the auxiliary reference frame is:

$$E_{(r)} = \frac{\partial}{\partial r}, \quad E_{(\varphi)} = \frac{1}{R \sin(\frac{r}{R})} \frac{\partial}{\partial \varphi}, \quad (196)$$

$$E_{(r)} = \frac{\partial}{\partial r}, \quad E_{(\varphi)} = \frac{1}{R \sinh(\frac{r}{R})} \frac{\partial}{\partial \varphi}, \quad (197)$$

respectively, in the spherical and pseudospherical case. These frames are evidently non-holonomic. The kinetic energy may be written in the form:

$$T = \frac{m}{2} G_{ij} \dot{q}^i \dot{q}^j, \quad (198)$$

where q^i are six generalized coordinates. We introduce them below; obviously, (φ, r) will be two of them. Now we can introduce the canonical formalism: $H = T + V$, where $T = \frac{1}{2m} G^{ij} p_i p_j$. The matrix G^{jk} is reciprocal to G_{ij} , i.e. $G_{ij} G^{jk} = \delta_i^k$. Translational kinetic energies have the form:

$$\text{sphere :} \quad T_{tr} = \frac{m}{2} \left(\left(\frac{dr}{dt} \right)^2 + R^2 \sin^2\left(\frac{r}{R}\right) \left(\frac{d\varphi}{dt} \right)^2 \right), \quad (199)$$

$$\text{pseudosphere :} \quad T_{tr} = \frac{m}{2} \left(\left(\frac{dr}{dt} \right)^2 + R^2 \sinh^2\left(\frac{r}{R}\right) \left(\frac{d\varphi}{dt} \right)^2 \right). \quad (200)$$

When the body is materially isotropic, in the two-dimensional case the inertia tensor has only one essential component $J_1 = J_2 = J$. It is convenient to use the two-polar decomposition $\varphi = UDV^{-1}$, where U, V are orthogonal and D -is diagonal. Generalized internal coordinates are $\alpha, \beta, \lambda, \mu$:

$$U = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}, \quad D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},$$

$$V = \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}.$$

Then T_{int} is given by:

$$\begin{aligned} T_{\text{int}} &= \frac{J}{2} \left[\left(\frac{d\lambda}{dt} \right)^2 + \left(\frac{d\mu}{dt} \right)^2 \right] + \frac{J}{2} [\lambda^2 + \mu^2] \chi^2 \\ &+ \frac{J}{2} [\lambda^2 + \mu^2] \vartheta^2 + 2J\lambda\mu\chi\vartheta, \end{aligned} \quad (201)$$

where, $\vartheta = \frac{d\beta}{dt}$ and χ is given by

$$\chi = \frac{d\alpha}{dt} + \cos\left(\frac{r}{R}\right) \frac{d\varphi}{dt} \quad (202)$$

on the 2-dimensional sphere and

$$\chi = \frac{d\alpha}{dt} + \cosh\left(\frac{r}{R}\right) \frac{d\varphi}{dt} \quad (203)$$

on the 2-dimensional pseudosphere. It is convenient to introduce new variables $\gamma := \alpha + \beta$, $\delta := \alpha - \beta$, $x := \frac{1}{\sqrt{2}}(\lambda - \mu)$, $y := \frac{1}{\sqrt{2}}(\lambda + \mu)$. Then in the spherical case $T = \frac{m}{2} G_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt}$, where $[\dots q^i \dots] = [r, \varphi, \gamma, \delta, x, y]$ and the matrix $[G_{ij}]$ is given by

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & [\tilde{G}_{kl}] & 0 & 0 \\ 0 & 0 & \frac{J}{m} & 0 \\ 0 & 0 & 0 & \frac{J}{m} \end{bmatrix}, \quad (204)$$

and the 3×3 block $[\tilde{G}_{kl}]$ is given by

$$[\tilde{G}_{kl}] = \begin{bmatrix} R^2 \sin^2\left(\frac{r}{R}\right) + \frac{J \cos^2\left(\frac{r}{R}\right)(x^2 + y^2)}{m} & \frac{Jx^2 \cos\left(\frac{r}{R}\right)}{m} & \frac{Jy^2 \cos\left(\frac{r}{R}\right)}{m} \\ \frac{Jx^2 \cos\left(\frac{r}{R}\right)}{m} & \frac{Jx^2}{m} & 0 \\ \frac{Jy^2 \cos\left(\frac{r}{R}\right)}{m} & 0 & \frac{Jy^2}{m} \end{bmatrix}.$$

In the canonical formalism: $T = \frac{1}{2m} G^{ij} p_i p_j$, $H = T - V$, where $G_{ij} G^{jk} = \delta_i^k$. One can show that if generalized coordinates are ordered as $[r, \varphi, \gamma, \delta, x, y]$, then the matrix $[G^{ab}]$ is given by

$$[G^{ab}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & [\tilde{G}^{cd}] & 0 & 0 \\ 0 & 0 & \frac{J}{m} & 0 \\ 0 & 0 & 0 & \frac{J}{m} \end{bmatrix}, \quad (205)$$

where

$$[\tilde{G}^{cd}] = \begin{bmatrix} \frac{1}{R^2 \sin^2(\frac{r}{R})} & \frac{-\cos(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} & \frac{-\cos(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} \\ \frac{-\cos(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} & \frac{m}{Jx^2} + \frac{-\cos^2(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} & \frac{\cos^2(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} \\ \frac{-\cos(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} & \frac{\cos^2(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} & \frac{m}{Jy^2} + \frac{-\cos^2(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} \end{bmatrix}.$$

In the pseudospherical case the analogous equations are valid. The only difference is that trigonometric functions are replaced by (properly signed) hyperbolic functions. So we have $T = \frac{m}{2} G_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt}$, where $[\dots q^i \dots] = [r, \varphi, \gamma, \delta, x, y]$ and the matrices $[G_{ij}]$ and $[G^{ab}]$ are given in the same form as in spherical problem (204), (205). We must only replace function $\sin \frac{r}{R}$ by $\sinh \frac{r}{R}$ and $\cos \frac{r}{R}$ by $\cosh \frac{r}{R}$. In the canonical formalism it is similarly. The generalized coordinates are ordered as $[r, \varphi, \gamma, \delta, x, y]$, r is now "pseudospherical radius" and $r \in (0 \dots \infty)$. Explicitly the kinetic term in the Hamiltonian on the sphere has the form

$$\begin{aligned} T &= \frac{p_r^2}{2m} + \frac{p_\varphi^2 - 2 \cos \frac{r}{R} p_\varphi (p_\gamma + p_\delta)}{2mR^2 \sin^2 \frac{r}{R}} \\ &+ \frac{\left(\frac{mR^2}{J} \sin^2 \frac{r}{R} + \cos^2 \frac{r}{R} \right) (p_\gamma + p_\delta)^2}{2mR^2 \sin^2 \frac{r}{R}} \\ &+ \frac{p_x^2}{2J} + \frac{p_y^2}{2J} + \frac{p_\gamma^2}{2Jx^2} + \frac{p_\delta^2}{2Jy^2}, \end{aligned} \quad (206)$$

and on the pseudosphere

$$\begin{aligned} T &= \frac{p_r^2}{2m} + \frac{p_\varphi^2 - 2 \cosh \frac{r}{R} p_\varphi (p_\gamma + p_\delta)}{2mR^2 \sinh^2 \frac{r}{R}} \\ &+ \frac{\left(\cosh^2 \frac{r}{R} \pm \frac{mR^2}{J} \sinh^2 \frac{r}{R} \right) (p_\gamma + p_\delta)^2}{2mR^2 \sinh^2 \frac{r}{R}} \\ &+ \frac{p_x^2}{2J} + \frac{p_y^2}{2J} + \frac{p_\gamma^2}{2Jx^2} + \frac{p_\delta^2}{2Jy^2}. \end{aligned} \quad (207)$$

The Hamilton-Jacobi equation: $\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0$ will be again reduced by the substitution: $S = -Et + S_0(q)$. The time-independent Hamilton-Jacobi equation has the form $H\left(q, \frac{\partial S_0}{\partial q}\right) = E$, and we seek for solutions of the form $S_0 = S_r(r) + S_\varphi(\varphi) + S_\gamma(\gamma) + S_\delta(\delta) + S_x(x) + S_y(y) = S_r(r) + l\varphi + C_\gamma\gamma + C_\delta\delta + S_x(x) + S_y(y)$. We can separate these equations if φ, γ and δ are cyclic variables. It is more convenient if we put $p_\gamma = \frac{1}{2}(p_\alpha + p_\beta)$ and $p_\delta = \frac{1}{2}(p_\alpha - p_\beta)$. Then: $S_0 = S_r(r) + S_\varphi(\varphi) + S_\gamma(\gamma) + S_\delta(\delta) + S_x(x) + S_y(y) = S_r(r) + l\varphi + C_\alpha\alpha +$

$C_\beta\beta + S_x(x) + S_y(y)$. In the spherical case the Hamilton-Jacobi equation has the form:

$$\begin{aligned}
E &= \frac{1}{2m} \left(\frac{dS_r(r)}{dr} \right)^2 + \frac{(l - C_\alpha \cos \frac{r}{R})^2}{2mR^2 \sin^2 \frac{r}{R}} + V(r) \\
&+ \frac{1}{2J} \left(\frac{dS_x}{dx} \right) + \frac{(C_\alpha + C_\beta)^2}{8Jx^2} + V_x(x) \\
&+ \frac{1}{2J} \left(\frac{dS_y}{dy} \right) + \frac{(C_\alpha - C_\beta)^2}{8Jy^2} + V_y(y).
\end{aligned} \tag{208}$$

In the pseudospherical case:

$$\begin{aligned}
E &= \frac{1}{2m} \left(\frac{dS_r(r)}{dr} \right)^2 + \frac{(l - C_\alpha \cosh \frac{r}{R})^2}{2mR^2 \sinh^2 \frac{r}{R}} + V(r) \\
&+ \frac{1}{2J} \left(\frac{dS_x}{dx} \right) + \frac{(C_\alpha + C_\beta)^2}{8Jx^2} + V_x(x) \\
&+ \frac{1}{2J} \left(\frac{dS_y}{dy} \right) + \frac{(C_\alpha - C_\beta)^2}{8Jy^2} + V_y(y).
\end{aligned} \tag{209}$$

Now we can calculate the action variables. For the all considered cases α, β and δ are cyclic variables and the corresponding actions have the same form:

$$\begin{aligned}
J_\varphi &= \oint \frac{dS_\varphi(\varphi)}{d\varphi} d\varphi = \int_0^{2\pi} l d\varphi = 2\pi l \Rightarrow l = \frac{J_\varphi}{2\pi}, \\
J_\alpha &= \oint \frac{dS_\alpha(\alpha)}{d\alpha} d\alpha = C_\alpha \int_0^{2\pi} d\alpha = 2\pi C_\alpha \Rightarrow C_\alpha = \frac{J_\alpha}{2\pi}, \\
J_\beta &= \oint \frac{dS_\beta(\beta)}{d\beta} d\beta = C_\beta \int_0^{2\pi} d\beta = 2\pi C_\beta \Rightarrow C_\beta = \frac{J_\beta}{2\pi}.
\end{aligned} \tag{210}$$

Now we can simplify the Hamilton-Jacobi equation using $J_\varphi, J_\alpha, J_\beta$. After this we can calculate J_r, J_x, J_y . The constants of separation C_x and C_y have the same form for the sphere and pseudosphere cases:

$$C_x := \frac{1}{2J} \left(\frac{dS_x}{dx} \right) + \frac{(J_\alpha + J_\beta)^2}{32\pi^2 J x^2} + V_x(x), \tag{211}$$

$$C_y := \frac{1}{2J} \left(\frac{dS_y}{dy} \right) + \frac{(J_\alpha - J_\beta)^2}{32\pi^2 J y^2} + V_y(y). \tag{212}$$

1. On the sphere we obtain:

$$\begin{aligned}
J_r &= \oint \sqrt{2m(E - C_x - C_y - V_r(r)) - \frac{(J_\varphi - J_\alpha \cos \frac{r}{R})^2}{4\pi^2 R^2 \sin^2 \frac{r}{R}}} dr, \\
J_x &= \oint \sqrt{2J(C_x - V_x(x)) - \frac{(J_\alpha + J_\beta)^2}{16\pi^2 x^2}} dx, \\
J_y &= \oint \sqrt{2J(C_y - V_y(y)) - \frac{(J_\alpha - J_\beta)^2}{16\pi^2 y^2}} dy.
\end{aligned} \tag{213}$$

2. On the pseudosphere the formulas read:

$$\begin{aligned}
J_r &= \oint \sqrt{2m(E - C_x - C_y - V_r(r)) - \frac{(J_\varphi - J_\alpha \cosh \frac{r}{R})^2}{4\pi^2 R^2 \sinh^2 \frac{r}{R}}} dr, \\
J_x &= \oint \sqrt{2J(C_x - V_x(x)) - \frac{(J_\alpha + J_\beta)^2}{16\pi^2 x^2}} dx, \\
J_y &= \oint \sqrt{2J(C_y - V_y(y)) - \frac{(J_\alpha - J_\beta)^2}{16\pi^2 y^2}} dy.
\end{aligned} \tag{214}$$

Substituting (210) into (210), (214) and expressions for C_x and C_y one obtains the explicit dependence of E on the action variables. The Hamilton-Jacobi equation is separable for some realistic (in the elasticity theory sense) potentials of the form $\mathfrak{U}(q) = V_r(r) + V_x(x) + V_y(y)$ both on the sphere and pseudosphere.

4.1 Models of potentials

4.1.1 Models of potentials in the "deformations" plane

There exists an interesting class of "universally" separable potentials. The corresponding Hamilton-Jacobi equations are separable in all coordinate systems used below, i.e., Cartesian coordinates (x, y) and polar coordinates (ς, ε) in the (x, y) -plane. Among those potentials there are ones effectively integrable, realistic, and applicable in elastic problems concerning internal degrees of freedom. Some of them are also separable in elliptic coordinates in the (x, y) -plane, however, they are rather non-useful in elastic problems, so we do not make use of elliptic coordinates and corresponding action variables. The mentioned potentials have the general form:

$$V(x, y) = \frac{A}{x^2} + \frac{B}{y^2} + C(x^2 + y^2), \tag{215}$$

where A, B, C are constants. An interesting and applicable subclass is given by:

$$V(x, y) = \frac{F}{y^2} + \frac{F}{4}(x^2 + y^2), \quad (216)$$

F is constant. The action variables are then given by

$$J_x = -\sqrt{\frac{(J_\alpha + J_\beta)^2}{16}} - \pi C_x \sqrt{\frac{2I}{F}}, \quad J_y = -\pi C_y \sqrt{\frac{2I}{F}} - \pi \sqrt{2IF + \frac{(J_\alpha - J_\beta)^2}{16\pi^2}}.$$

Another class of separable potentials very well suited to nonlinear elastic problems is given by expressions:

$$V(\varsigma, \varepsilon) = V_\varsigma(\varsigma) + \frac{V_\varepsilon(\varepsilon)}{\varsigma^2 \sin^2(\varepsilon)},$$

where (ς, ε) are polar coordinates in the (x, y) -plane of deformation invariants:

$$x = \varsigma \sin \varepsilon, \quad y = \varsigma \cos \varepsilon, \quad (217)$$

$V_\varsigma(\varsigma), V_\varepsilon(\varepsilon)$ are functions of the one indicated variable, respectively ς, ε . Appropriately choosing these "shape functions" we can just obtain the mentioned compatibility with the standard requirements of nonlinear elastic dynamics. In spherical case (we mean here spherical geometry of two-dimensional "physical" space) when generalized coordinates are ordered as $[r, \varphi, \alpha, \beta, \varepsilon, \varepsilon]$, the metric $[G^{ij}]$, has the form

$$[G^{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & [\tilde{G}^{kl}] & 0 & 0 \\ 0 & 0 & \frac{m}{\varsigma^2 J} & 0 \\ 0 & 0 & 0 & \frac{m}{J} \end{bmatrix} \quad (218)$$

and

$$[\tilde{G}^{kl}] = \begin{bmatrix} \frac{1}{R^2 \sin^2(\frac{r}{R})} & \frac{-\cos(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} & \frac{-\cos(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} \\ \frac{-\cos(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} & \wp + \frac{\cot^2(\frac{r}{R})}{R^2} & \frac{1}{R^2} \cot^2\left(\frac{r}{R}\right) \\ \frac{-\cos(\frac{r}{R})}{R^2 \sin^2(\frac{r}{R})} & \frac{1}{R^2} \cot^2\left(\frac{r}{R}\right) & \wp + \frac{\cot^2(\frac{r}{R})}{R^2} \end{bmatrix}$$

, where

$$\wp := \frac{m}{J(\varsigma \cos \varepsilon)^2}. \quad (219)$$

The separation constants C_x, C_y will be combined into a single one,

$$C = -C_x - C_y \quad (220)$$

and

$$\begin{aligned}
C &= \frac{1}{2J} \left(\frac{\partial S_\zeta}{\partial \zeta} \right)^2 + \frac{1}{2J\zeta^2} \left(\frac{\partial S_\varepsilon}{\partial \varepsilon} \right)^2 \\
&+ \frac{(J_\alpha + J_\beta)^2}{8\pi^2 J\zeta^2 (\sin 2\varepsilon)} + V_{\varepsilon\zeta}(\zeta, \varepsilon).
\end{aligned} \tag{221}$$

The new constant of separation A appears

$$A := \frac{1}{2J} \left(\frac{\partial S_\varepsilon}{\partial \varepsilon} \right)^2 \frac{(J_\alpha + J_\beta)^2}{8\pi^2 J (\sin 2\varepsilon)} + V_\varepsilon(\varepsilon), \tag{222}$$

$$J_\varepsilon = \oint \sqrt{2J(A - V_\varepsilon(\varepsilon)) - \frac{(J_\alpha^2 + 2\cos(2\varepsilon)J_\alpha J_\beta + J_\beta^2)}{4\pi^2 R^2 \sin^2(2\varepsilon)}} d\varepsilon, \tag{223}$$

$$J_\zeta = \oint \sqrt{2I(C - V_\zeta(\zeta)) - \frac{2JA}{\zeta^2}} d\zeta. \tag{224}$$

As $V_\varepsilon(\varepsilon)$ we propose:

$$V_\varepsilon(\varepsilon) = \hat{\gamma} \cot^2(2\varepsilon),$$

where $\hat{\gamma}$ is some constant, then:

$$J_\varepsilon = \frac{1}{4} \left(4\sqrt{2J(A + \hat{\gamma})}\pi - \sqrt{8J\hat{\gamma}\pi^2 + (J_\alpha - J_\beta)^2} - \sqrt{8J\hat{\gamma}\pi^2 + (J_\alpha + J_\beta)^2} \right).$$

As $V_\zeta(\zeta)$ let us take

$$V_\zeta(\zeta) = \frac{\tilde{\gamma}}{\zeta},$$

where $\tilde{\gamma}$ is some constant, then:

$$J_\zeta = \sqrt{2} \left(-2\sqrt{JA} + \frac{\sqrt{J}\tilde{\gamma}}{\sqrt{C_x + C_y}} \right) \pi.$$

4.1.2 Models for the potential $V_r(r)$

The next important problem is to suggest some physically interesting and computationally effective models for the "radial" potentials $V_r(r)$ in the physical space. One can expect that computationally effective will be potentials somehow suited to geometry of the "physical" space M . It is well-known that the determinant $\det[g^{ij}]$ is a scalar density of weight -2 . This is an important geometric object, used, e.g., for expressing the two-dimensional "volume" (surface area) element. Namely, this element is analytically given by

$$d\mu(r, \varphi) \frac{1}{\sqrt{\det[g^{ij}]}} dr d\varphi = \sqrt{\det[g_{ij}]} dr d\varphi. \tag{225}$$

One can expect that simple expressions built of $\det[g^{ij}]$ will be good candidates for the radial potentials, both mechanically interesting and computationally effective. We assume the general form:

$$V_r(r) = f(r)\det[g^{ij}]. \quad (226)$$

f being something "simple". Obviously, it is constant that is the simplest; we denote it by

$$f = R^2\gamma,$$

γ being also some constant, and then in spherical case where $\det[g_{ij}] = \frac{1}{R^2 \sin^2(\frac{r}{R})}$ we have potential $V_r(r)$ in the form:

$$V_r(r) = \frac{\gamma}{\sin^2(\frac{r}{R})}.$$

The "radial" action variable J_r (210) is then given by:

$$\begin{aligned} J_r &= \sqrt{2m4\pi^2 R^2(E+C) + J_\alpha^2} \\ &- \frac{1}{2}\sqrt{2m4\pi^2 R^2\gamma + (J_\varphi + J_\alpha)^2} \\ &- \frac{1}{2}\sqrt{2m4\pi^2 R^2\gamma + (J_\varphi - J_\alpha)^2}. \end{aligned} \quad (227)$$

With $-C_x - C_y = C$ substituted from (221). Obviously, in the compact spherical manifold all translational motion are bounded on the "physical" space, therefore, the situation where $\gamma = 0$, i.e., $V_r(r) = 0$ are admissible for the existence of well-defined action variables J_r . then we have

$$\begin{aligned} J_r &= \sqrt{2m4\pi^2 R^2(E - C_x - C_y) + J_\alpha^2} \\ &- \frac{1}{2}|J_\varphi - J_\alpha| - \frac{1}{2}|J_\varphi + J_\alpha| = |J_\varphi| \geq |J_\alpha| \\ &= \sqrt{2m4\pi^2 R^2(E - C_x - C_y) + J_\alpha^2} - J_\varphi. \end{aligned} \quad (228)$$

In pseudospherical case

$$\det[g_{ij}] = \frac{1}{R^2 \sinh^2(\frac{r}{R})}$$

and again we put

$$f(r) = \gamma,$$

γ being constant, and then

$$V(r) = \frac{\gamma}{R^2 \sinh^2(\frac{r}{R})},$$

This also leads to some explicitly integrable expression for J_r (214). However negative if motion in the r -variable is to be bounded and the corresponding action variable J_r to be finite.

$$\begin{aligned} J_r &= -\sqrt{2m4\pi^2 R^2(C_x + C_y - E) + J_\alpha^2} \\ &+ \frac{1}{2}\sqrt{2m4\pi^2 \gamma + (J_\varphi + J_\alpha)^2} \\ &- \frac{1}{2}\sqrt{2m4\pi^2 \gamma + (J_\varphi - J_\alpha)^2}. \end{aligned}$$

Solving the formulas (227), (228) with respect to E , we express the energy as a function of all action variables. For that C_x, C_y must be also expressed by action variables. One achieves this by solving (210) with respect to C_x, C_y and substituting the resulting expressions to (228), (228).

5 Some additional remarks

We have mentioned above that there exist some models, interesting at least from purely geometric point of view of analytical mechanics in itself. This has to do with our models of affinely invariant dynamics as developed in [48], [49], [50]. Namely, we can assume in M only some affine connection structure (M, Γ) without any fixed metric tensor g on M . The role of instantaneous metric tensor at the configuration $e \in F_x M \subset FM$ will be played by the Cauchy deformation tensor; analytically

$$C[e]_{ij} = \eta_{AB} e^A_i e^B_j, \quad (229)$$

where η denotes the micromaterial metric. One can always choose coordinates in such a way that $\eta_{AB} = \delta_{AB}$ (if η is positively definite, what is physically assumed). And further on we postulate kinetic energy in the "usual" form:

$$T = \frac{m}{2} C_{ij}[e] \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{1}{2} C_{ij}[e] \left(\frac{D}{Dt} e^i_A \right) \left(\frac{D}{Dt} e^j_A \right) J^{AB} \quad (230)$$

$$= \frac{m}{2} \eta_{AB} \hat{v}^A \hat{v}^B + \frac{1}{2} \eta_{AB} \hat{\Omega}^A_C \hat{\Omega}^B_D J^{CD}, \quad (231)$$

where \hat{v}^A are co-moving components of translational velocity. After some calculations based on Poisson brackets we obtain equations of motion in the following balance form:

$$m \frac{Dv^a}{Dt} = m (\Omega^a_b + \Omega^a_b) v^b + 2m v^b v^c S_{cb}^a + \Sigma^c_d R^d_{cab} v^b + F^a, \quad (232)$$

$$m \frac{D\Sigma^{ab}}{Dt} = \Sigma^{ac} (\Omega^b_c + \Omega^b_c) - m v^a v^b + N^{ab}, \quad (233)$$

where indices are raised and lowered with the help of C as a "metric tensor". In the purely covariant and mixed tensor form, these equations acquire the following suggestive shape:

$$\frac{Dp_a}{Dt} = m \frac{Dv_a}{Dt} = 2mv^b v^c S_{cba} + \Sigma^c_d R^d_{cab} v^b + F^a, \quad (234)$$

$$m \frac{D\Sigma^a_b}{Dt} = N^a_b - mv^a v_b, \quad (235)$$

with the connection concerning the shift of indices. These formulas are very similar formally to (156), however we must remember about important difference: no metric field g is now defined all over M . The Cauchy "metric" $C[e]$ is defined only at the point $x = \pi(e)$ where the object is instantaneously present. Although no metric field g is fixed in M , nevertheless we can introduce the concept of rigid motion by putting the following conditions on motion of our object:

$$\frac{DC_{ij}}{Dt} = 0, \quad (236)$$

and this is simply equivalent to:

$$\Omega^i_j = -\Omega_j^i = -C_{jk} C^{-1il} \Omega^k_l, \quad (237)$$

i.e., the C -skew-symmetry of Ω . It is interesting that unlike the usual g -skew-symmetry of Ω , i.e., the g -metrical rigid motion, these constraints are in general non-holonomic.

To finish with let us remind our models of affinely-invariant dynamics (in the left- and rigid-sense) of extended affine bodies in flat spaces. There are natural counterparts manifolds. Here we merely quote the corresponding kinetic energies, analogous to ones introduced in [48], [49], [50].

For the spatially affine and micromaterially isotropic models we have:

$$\begin{aligned} T &= \frac{m}{2} \eta_{AB} \widehat{v}^A \widehat{v}^B + \frac{I}{2} \eta_{KL} \eta^{MN} \widehat{\Omega}^K_M \widehat{\Omega}^L_N + \frac{A}{2} \text{Tr}(\widehat{\Omega}^2) + \frac{B}{2} \text{Tr}(\widehat{\Omega})^2 \\ &= \frac{m}{2} C_{ij} v^i v^j + \frac{I}{2} C_{kl} C^{mn} \Omega^k_m \Omega^l_n + \frac{A}{2} \Omega^i_j \Omega^j_i + \frac{B}{2} \Omega^i_i \Omega^j_j. \end{aligned}$$

Similarly, for the spatially metrical and micromaterially affine models (discretization of the Arnold description of fluids) we have:

$$\begin{aligned} T &= \frac{m}{2} G_{AB} \widehat{v}^A \widehat{v}^B + \frac{I}{2} G_{KL} G^{MN} \widehat{\Omega}^K_M \widehat{\Omega}^L_N + \frac{A}{2} \text{Tr}(\widehat{\Omega}^2) + \frac{B}{2} \text{Tr}(\widehat{\Omega})^2 \\ &= \frac{m}{2} g_{ij} v^i v^j + \frac{I}{2} g_{kl} g^{mn} \Omega^k_m \Omega^l_n + \frac{A}{2} \Omega^i_j \Omega^j_i + \frac{B}{2} \Omega^i_i \Omega^j_j. \end{aligned}$$

Obviously, in spite of this redundant way of writing, we remember that $\text{Tr}(\Omega^2) = \text{Tr}(\widehat{\Omega}^2)$, $\text{Tr}(\Omega) = \text{Tr}(\widehat{\Omega})$. Models of this kind were analyzed in flat space motion [48], [49], [50]. In curved manifolds the problem is much more complicated

and will be analysed only in some special cases. The more detailed study is postponed to the later papers.

Acknowledgment The research presented above was supported by the Ministry of Science and Higher Education grant No 501 018 32/1992.

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